Improper integrals (Sect. 8.7)

- Review: Improper integrals type I and II.
- Examples: $I = \int_1^{\infty} \frac{dx}{x^p}$, and $I = \int_0^1 \frac{dx}{x^p}$.
- Convergence test: Direct comparison test.
- Convergence test: Limit comparison test.

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If f is continuous on $[a, c) \cup (c, b]$ and discontinuous at c, then

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The cases $\displaystyle\int_0^1 \frac{dx}{x^p}$ and $\displaystyle\int_1^\infty \frac{dx}{x^p}$

Summary: In the case $p = 1$ both integrals diverge,

$$\int_0^1 \frac{dx}{x} = \text{diverges}, \qquad \int_1^\infty \frac{dx}{x} = \text{diverges}.$$

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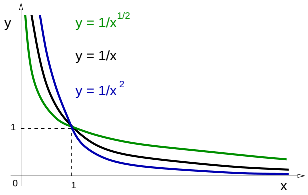

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- Convergence test: Limit comparison test.

Convergence test: Direct comparison test

Remark: Convergence tests determine whether an improper integral converges or diverges.

Theorem (Direct comparison test)

If functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and $0 \leqslant f(x) \leqslant g(x)$ for every $x \in [a, \infty)$, then holds

$$0 \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx.$$

The inequalities above imply the following statements:

(a) $\displaystyle\int_a^\infty g(x)\,dx$ *converges* $\Rightarrow$ $\displaystyle\int_a^\infty f(x)\,dx$ *converges;*

(b) $\displaystyle\int_a^\infty f(x)\,dx$ *diverges* $\Rightarrow$ $\displaystyle\int_a^\infty g(x)\,dx$ *diverges.*

Convergence test: Direct comparison test

Example

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The last inequality follows because exp is an increasing function.

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Since $0 \leqslant \int_1^{\infty} e^{-x^2}\,dx \leqslant \frac{1}{e}$, the integral converges. $\triangleleft$

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Since $0 \leqslant \int_1^\infty \frac{dx}{\sqrt{x^6+1}} \leqslant \frac{1}{2}$, the integral converges. $\lhd$

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- **Convergence test: Limit comparison test.**

Convergence test: Limit comparison test

Remark: Convergence tests determine whether an improper integral converges or diverges.

Theorem (Limit comparison test)

If positive functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \textit{with} \quad 0 < L < \infty,$$

then the integrals

$$\int_a^\infty f(x)\,dx, \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

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Remark: Although both integrals above may converge, their values need **not** be the same.

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Determine whether $I = \int_1^{\infty} \frac{dx}{\sqrt{x^6 + 1}}$ converges or diverges.

Solution: The convergence of integrals involving rational functions is simple to determine with the limit comparison test.

Convergence test: Limit comparison test

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Since $\int_1^{\infty} \frac{dx}{x^3}$ converges, then $\int_1^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges too. $\triangleleft$

Convergence test: Limit comparison test

Example

Determine whether $I = \int_3^\infty \frac{dx}{(2+\cos(x)+\ln(x))}$ converges or not.

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The limit comparison test says:

Convergence test: Limit comparison test

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Since $\int_3^{\infty} \frac{dx}{x}$ diverges,

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Since $\int_3^\infty \frac{dx}{x}$ diverges, then both J and I diverge. $\triangleleft$

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Determine whether $I = \int_3^\infty \frac{x\,dx}{\sqrt{x^5 + x^3}}$ converges or not.

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we conclude that I converges. ◁

Infinite sequences (Sect. 10.1)

Today's Lecture:

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- The Sandwich Theorem for sequences.

Next Lecture:

- The Continuous Function Theorem for sequences.
- Using L'Hôpital's rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

Overview: Sequences, series, and calculus

Remarks:

- We have defined the $\int_a^b f(x)\,dx$ as a limit of partial sums.

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- We will express differentiable functions as infinite sums of polynomials (Taylor series expansions).
- Then we will be able to compute integrals like $\int_a^b e^{-x^2}\,dx$.

Infinite sequences (Sect. 10.1)

- Overview: Sequences, series, and calculus.
- **Definition and geometrical representations.**
- The limit of a sequence, convergence, divergence.
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Definition and geometrical representations

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Definition and geometrical representations

Example

Find a formula for the general term of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \dots\right\}.$$

Definition and geometrical representations

Example

Find a formula for the general term of the sequence

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Solution: We know that:

$$a_1 = \frac{3}{5},$$

Definition and geometrical representations

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We conclude that $a_n = (-1)^{(n-1)} \dfrac{(n+2)}{5^n}$. ◁

Definition and geometrical representations

Remark:

Infinite sequences can be represented on a line or on a plane.

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Graph the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ on a line and on a plane.

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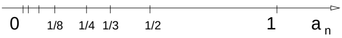

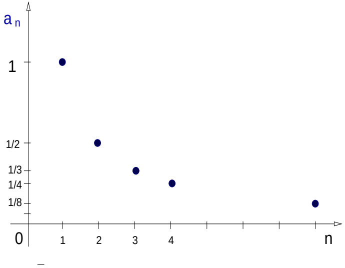

Infinite sequences (Sect. 10.1)

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- **The limit of a sequence, convergence, divergence.**
- Properties of sequence limits.
- The Sandwich Theorem for sequences.

The limit of a sequence, convergence, divergence

Remark:

- As it happened in the example above, the numbers a_n in a sequence may approach a single value as n increases.

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The limit of a sequence, convergence, divergence

Definition

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Solution: Since $\dfrac{1}{n^2} \to 0$, we will prove that $L = 1$.

The limit of a sequence, convergence, divergence

Example

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Properties of sequence limits

Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.

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Theorem (Limit properties)

If the sequence $\{a_n\} \to A$ *and* $\{b_n\} \to B$, *then holds,*

(a) $\lim_{n\to\infty} \{a_n + b_n\} = A + B$;

(b) $\lim_{n\to\infty} \{a_n - b_n\} = A - B$;

(c) $\lim_{n\to\infty} \{k a_n\} = kA$;

(d) $\lim_{n\to\infty} \{a_n b_n\} = AB$;

(e) *If* $B \neq 0$, *then* $\lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$.

Properties of sequence limits

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Find the limit of the sequence $\left\{a_n = \dfrac{1-2n}{2+3n}\right\}_{n=1}^{\infty}$.

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Hence, the quotient property implies $a_n \to -\dfrac{2}{3}$. ◁

Properties of sequence limits

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Solution: Rewrite the sequence as follows,

$$a_n = \frac{(3n^3 - 2n + 1)}{(2n^2 + 4)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{3n - \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{4}{n^2}}$$

Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, then

$$\frac{1}{n^2} = \left(\frac{1}{n}\right)^2 \to 0, \qquad \frac{2}{n} \to 0, \qquad 2 + \frac{4}{n^2} \to 2.$$

Hence, the quotient property implies $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{2}$.

Properties of sequence limits

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We conclude that a_n diverges. ◁

Infinite sequences (Sect. 10.1)

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- **The Sandwich Theorem for sequences.**

The Sandwich Theorem for sequences

Theorem (Sandwich-Squeeze)

If the sequences $\{a_n\}$, $\{b_n\}$, *and* $\{c_n\}$ *satisfy*

$$a_n \leqslant b_n \leqslant c_n, \quad \textit{for} \quad n > N,$$

and if both $a_n \to L$ *and* $c_n \to L$, *then holds*

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Since $\pm \dfrac{1}{n^2} \to 0$, we conclude that $a_n \to 0$. ◁

Infinite sequences (Sect. 10.1)

Today's Lecture:

- Review: Infinite sequences.
- The Continuous Function Theorem for sequences.
- Using L'Hôpital's rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

Previous Lecture:

- Overview: Sequences, series, and calculus.
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Review: Infinite sequences

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An infinite sequence of numbers is an ordered set of real numbers.

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An infinite sequence $\{a_n\}$ has limit L iff for every number $\epsilon > 0$ there exists a positive integer N such that

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Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.

Infinite sequences (Sect. 10.1)

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The Continuous Function Theorem for sequences

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If a sequence $\{a_n\} \to L$ *and a continuous function* f *is defined both at* L *and every* a_n*, then the sequence* $\{f(a_n)\} \to f(L)$.

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Example

Find the limit of $\left\{ \ln \left(\frac{(2 + n + 3n^2)}{(2n^2 + 3)} \right) \right\}$ as $n \to \infty$.

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We conclude that $b_n \to \ln\left(\dfrac{3}{2}\right)$. $\triangleleft$

Infinite sequences (Sect. 10.1)

- Review: Infinite sequences.
- The Continuous Function Theorem for sequences.
- **Using L'Hôpital's rule on sequences.**
- Table of useful limits.
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Using L'Hôpital's rule on sequences

Theorem (L'Hôpital's rule for sequences)

If the sequence $\{a_n\}$ *satisfies that:*

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Solution: Notice that $a_n = f(n)$ for $f(x) = \sqrt[8x]{5x}$.

Using L'Hôpital's rule on sequences

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Find the limit $a_n = \sqrt[8n]{5n}$ as $n \to \infty$.

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Using L'Hôpital's rule on sequences

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Example

Find the limit $a_n = \sqrt[8n]{5n}$ as $n \to \infty$.

Solution: Recall: $a_n = f(n)$ for $f(x) = \sqrt[8x]{5x}$.

$$\sqrt[8x]{5x} = e^{\ln(\sqrt[8x]{5x})} = e^{\left(\frac{\ln(5x)}{8x}\right)}$$

But $\displaystyle\lim_{x\to\infty} \frac{\ln(5x)}{8x}$ is indeterminate $\dfrac{\infty}{\infty}$. L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln(5x)}{8x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{8} = \lim_{x\to\infty} \frac{1}{8x} = 0.$$

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We conclude that $\sqrt[8n]{5n} \to 1$ as $n \to \infty$. $\triangleleft$

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Given positive numbers a, b, find the $\lim_{n \to \infty} \left(1 - \frac{b}{n}\right)^{an}$.

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We conclude that $\displaystyle\lim_{n\to\infty}\Big(1-\frac{b}{n}\Big)^{an} = e^{ab}$. ◁

Infinite sequences (Sect. 10.1)

- Review: Infinite sequences.
- The Continuous Function Theorem for sequences.
- Using L'Hôpital's rule on sequences.
- **Table of useful limits.**
- Bounded and monotonic sequences.

Table of useful limits

Remark: The following limits appear often in applications:

- $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0;$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1;$
- $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1, \text{ for } x > 0;$
- $\lim_{n\to\infty} x^n = 0, \text{ for } |x| < 1;$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \text{ for } x \in \mathbb{R};$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

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A sequence $\{a_n\}$ is bounded above iff there is $M \in \mathbb{R}$ such that

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