Integrating using tables (Sect. 8.5)

- Remarks on:
  - Using Integration tables.
  - Reduction formulas.
  - Computer Algebra Systems.
  - Non-elementary integrals.
- Limits using L'Hôpital's Rule (Sect. 7.5).

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where we used that $x > 0$. Notice that the denominator does not vanishes for $x > 0$. After looking for a while in the integration tables at the end of the textbook, we find the entry (13b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right| + c.$$

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We can use this formula for $a = 2$ and $b = 3$.

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We can use this formula for $a = 2$ and $b = 3$. We conclude that

$$\int \frac{dx}{x\sqrt{2x+3}} = \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{2x+3} - \sqrt{3}}{\sqrt{2x+3} + \sqrt{3}}\right| + c.$$

◁

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Entry (15) in the integration tables at the end of the textbook is

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This formula relates a complicated integral to a simpler integral.

$$\int \frac{dx}{x^2\sqrt{(4x+9)}} = -\frac{\sqrt{4x+9}}{9x} - \frac{2}{9}\int \frac{dx}{x\sqrt{4x+9}}.$$

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We now use the entry (13b) again,

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and we get

$$I = -\frac{\sqrt{4x+9}}{9x} - \frac{2}{9} \left[\frac{1}{3} \ln \left| \frac{\sqrt{4x+9} - 3}{\sqrt{4x+9} + 3} \right| \right] + c. \qquad \triangleleft$$

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Solution: Maple gives:

$$I = \frac{x}{4}\,(a^2 + x^2)^{3/2} - \frac{a^2 x}{8}\,\sqrt{a^2 + x^2} - \frac{a^2}{8}\,\ln(x + \sqrt{a^2 + x^2}).$$

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Mathematica gives

$$\Big(\frac{a^2 x}{8} + \frac{x^3}{4}\Big)\,\sqrt{a^2 + x^2} - \frac{a^2}{8}\,\ln\big(x + \sqrt{a^2 + x^2}\big).$$

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Mathematica gives

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Both expressions define the same function. ◁

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- Example: $f(x) = \int \frac{dx}{x}$ is a new function. It is called $\ln(x)$.
- In a similar way, the following integrals define new functions:

$$\mathrm{erf} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt, \quad I_1 = \int \sin(x^2)\,dx, \quad I_2 = \int \frac{\sin(x)}{x}\,dx$$

$$I_2 = \int \sqrt{1+x^4}\,dx, \quad I_3 = \int \frac{e^x}{x}\,dx, \quad I_4 = \int \frac{dx}{\ln(x)}.$$

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Limits using L'Hôpital's Rule (Sect. 7.5)

Remarks:

- L'Hôpital's rule applies on limits of the form $L = \lim_{x \to a} \frac{f(x)}{g(x)}$ in the case that $f(a) = 0$ and $g(a) = 0$.

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Theorem

If functions $f, g : I \to \mathbb{R}$ are differentiable in an open interval containing $x = a$, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in I - \{a\}$, then holds

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right-hand side exists.

Limits using L'Hôpital's Rule (Sect. 7.5)

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The limit is indeterminate, $\dfrac{0}{0}$. But,

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We conclude $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1.$ ◁

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$$L = \lim_{x \to 0} \frac{(-1/4)(1+x)^{-3/2}}{2}$$

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$$L = \lim_{x\to 0} \frac{(-1/4)(1+x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$

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We conclude that $L = -\dfrac{1}{8}$. $\triangleleft$

Limits using L'Hôpital's Rule (Sect. 7.5)

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Evaluate $L = \lim_{x \to 0} \dfrac{x(1 - \cos(6x))}{(7x - \sin(7x))}$.

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$$L = \lim_{x \to 0} \frac{2(6)\sin(6x) + 6^2 x\cos(6x)}{7^2 \sin(7x)}$$

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This limit is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a third time,

$$L = \lim_{x \to 0} \frac{2(6^2)\cos(6x) + 6^2 \cos(6x) + 6^3 x \sin(6x)}{7^3 \cos(7x)}$$

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $L = \lim_{x \to 0} \dfrac{x(1 - \cos(6x))}{(7x - \sin(7x))}$.

Solution: Recall: $L = \lim_{x \to 0} \dfrac{2(6)\sin(6x) + 6^2 x \cos(6x)}{7^2 \sin(7x)}$.

This limit is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a third time,

$$L = \lim_{x \to 0} \frac{2(6^2)\cos(6x) + 6^2\cos(6x) + 6^3 x \sin(6x)}{7^3 \cos(7x)} = \frac{3(6^2)}{7^3}.$$

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $L = \lim_{x \to 0} \dfrac{x(1 - \cos(6x))}{(7x - \sin(7x))}$.

Solution: Recall: $L = \lim_{x \to 0} \dfrac{2(6)\sin(6x) + 6^2 x \cos(6x)}{7^2 \sin(7x)}$.

This limit is still indeterminate, $\dfrac{0}{0}$.

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We conclude that $L = \dfrac{3(6^2)}{7^3}$. ◁

Limits using L'Hôpital's Rule (Sect. 7.5)

- Review: L'Hôpital's rule for indeterminate limits $\frac{0}{0}$.
- Indeterminate limit $\frac{\infty}{\infty}$.
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- Overview of improper integrals (Sect. 8.7).

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Remarks:

- L'Hôpital's rule applies on limits of the form $L = \lim_{x \to a} \frac{f(x)}{g(x)}$ in the case that both $f(a) = 0$ and $g(a) = 0$.
- These limits are called indeterminate and denoted as $\frac{0}{0}$.

Theorem

If functions $f, g : I \to \mathbb{R}$ are differentiable in an open interval containing $x = a$, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in I - \{a\}$, then holds

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right-hand side exists.

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Example

Evaluate $L = \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$.

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Example

Evaluate $L = \lim_{x \to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\dfrac{0}{0}$.

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Example

Evaluate $L = \lim_{x \to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\frac{0}{0}$. But,

$$L = \lim_{x \to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x}.$$

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Example

Evaluate $L = \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$.

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The limit on the right-hand side is still indeterminate, $\frac{0}{0}$.

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Example

Evaluate $L = \lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

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The limit on the right-hand side is still indeterminate, $\frac{0}{0}$.

We use L'Hôpital's rule for a second time,

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

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We use L'Hôpital's rule for a second time,

$$L = \lim_{x \to 0} \frac{(-1/4)(1+x)^{-3/2}}{2}$$

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

Example

Evaluate $L = \lim_{x \to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$.

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The limit on the right-hand side is still indeterminate, $\frac{0}{0}$.

We use L'Hôpital's rule for a second time,

$$L = \lim_{x \to 0} \frac{(-1/4)(1+x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Example

Evaluate $L = \lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\dfrac{0}{0}$. But,

$$L = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x}.$$

The limit on the right-hand side is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a second time,

$$L = \lim_{x\to 0} \frac{(-1/4)(1+x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$

We conclude that $L = -\dfrac{1}{8}$. $\triangleleft$

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Remark: L'Hôpital's rule applies to indeterminate limits only.

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

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Example

Evaluate $L = \lim_{x \to 0} \dfrac{1 - \cos(x)}{x + x^2}$.

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

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Evaluate $L = \lim_{x \to 0} \dfrac{1 - \cos(x)}{x + x^2}$.

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Evaluate $L = \lim_{x \to 0} \dfrac{1 - \cos(x)}{x + x^2}$.

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L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

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$$L = \lim_{x \to 0} \frac{1 - \cos(x)}{x + x^2} = \lim_{x \to 0} \frac{\sin(x)}{1 + 2x} = \frac{0}{1} \quad \Rightarrow \quad L = 0. \qquad \triangleleft$$

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Remark:

- The limit $\frac{0}{1}$ is not indeterminate,

L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

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Remark:

- The limit $\frac{0}{1}$ is not indeterminate, since $\frac{0}{1} = 0$.

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Remark: L'Hôpital's rule applies to indeterminate limits only.

Example

Evaluate $L = \lim_{x\to 0} \dfrac{1-\cos(x)}{x+x^2}$.

Solution: The limit is indeterminate $\dfrac{0}{0}$. L'Hôpital's rule implies,

$$L = \lim_{x\to 0} \frac{1-\cos(x)}{x+x^2} = \lim_{x\to 0} \frac{\sin(x)}{1+2x} = \frac{0}{1} \quad \Rightarrow \quad L = 0. \qquad \triangleleft$$

Remark:

- The limit $\dfrac{0}{1}$ is not indeterminate, since $\dfrac{0}{1} = 0$.
- Therefore, L'Hôpital's rule does not hold in this case:

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

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Evaluate $L = \lim_{x\to 0} \dfrac{1-\cos(x)}{x+x^2}$.

Solution: The limit is indeterminate $\dfrac{0}{0}$. L'Hôpital's rule implies,

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$$\lim_{x\to 0} \frac{\sin(x)}{1+2x} \neq \lim_{x\to 0} \frac{\big(\sin(x)\big)'}{(1+2x)'}$$

L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$

Remark: L'Hôpital's rule applies to indeterminate limits only.

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Evaluate $L = \lim_{x\to 0} \dfrac{1-\cos(x)}{x+x^2}$.

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L'Hôpital's rule for indeterminate limits $\frac{0}{0}$

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Limits using L'Hôpital's Rule (Sect. 7.5)

- Review: L'Hôpital's rule for indeterminate limits $\frac{0}{0}$.
- **Indeterminate limit $\frac{\infty}{\infty}$.**
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- Overview of improper integrals (Sect. 8.7).

Indeterminate limit $\dfrac{\infty}{\infty}$

Remark: L'Hôpital's rule can be generalized to limits $\dfrac{\infty}{\infty}$, and also to side limits.

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Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \dfrac{2 + \tan(x)}{3 + \sec(x)}$.

Indeterminate limit $\frac{\infty}{\infty}$

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Example

Evaluate $L = \displaystyle\lim_{x \to (\frac{\pi}{2})^-} \frac{2 + \tan(x)}{3 + \sec(x)}$.

Solution: This is an indeterminate limit $\dfrac{\infty}{\infty}$. L'Hôpital's rule implies

$$\lim_{x \to (\frac{\pi}{2})^-} \frac{\big(2 + \tan(x)\big)'}{\big(3 + \sec(x)\big)'}$$

Indeterminate limit $\dfrac{\infty}{\infty}$

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Since $\dfrac{\sec(x)}{\tan(x)} = \dfrac{1}{\cos(x)} \dfrac{\cos(x)}{\sin(x)}$

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Since $\dfrac{\sec(x)}{\tan(x)} = \dfrac{1}{\cos(x)} \dfrac{\cos(x)}{\sin(x)} = \dfrac{1}{\sin(x)}$,

Indeterminate limit $\frac{\infty}{\infty}$

Remark: L'Hôpital's rule can be generalized to limits $\frac{\infty}{\infty}$, and also to side limits.

Example

Evaluate $L = \lim_{x\to(\frac{\pi}{2})^-} \frac{2+\tan(x)}{3+\sec(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\big(2+\tan(x)\big)'}{\big(3+\sec(x)\big)'} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$$

Since $\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x)}\frac{\cos(x)}{\sin(x)} = \frac{1}{\sin(x)}$, then $L = 1$. $\triangleleft$

Indeterminate limit $\frac{\infty}{\infty}$

Remark: Sometimes L'Hôpital's rule is not useful.

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Indeterminate limit $\frac{\infty}{\infty}$

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Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: We know that this limit can be computed simplifying:

Indeterminate limit $\frac{\infty}{\infty}$

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Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

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$$\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x)} \frac{\cos(x)}{\sin(x)}$$

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Indeterminate limit $\frac{\infty}{\infty}$

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Evaluate $L = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

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$$\frac{\sec(x)}{\tan(x)} = \frac{1}{\cos(x)}\frac{\cos(x)}{\sin(x)} = \frac{1}{\sin(x)} \quad \Rightarrow \quad L = 1. \qquad \triangleleft$$

We now try to compute this limit using L'Hôpital's rule.

Indeterminate limit $\dfrac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \dfrac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\dfrac{\infty}{\infty}$.

Indeterminate limit $\dfrac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \dfrac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\dfrac{\infty}{\infty}$. L'Hôpital's rule implies

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\sec(x))'}{(\tan(x))'}$$

Indeterminate limit $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

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Indeterminate limit $\frac{\infty}{\infty}$

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Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

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Indeterminate limit $\frac{\infty}{\infty}$

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Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

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The later limit is once again indeterminate, $\frac{\infty}{\infty}$.

Indeterminate limit $\dfrac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \dfrac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\dfrac{\infty}{\infty}$. L'Hôpital's rule implies

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The later limit is once again indeterminate, $\dfrac{\infty}{\infty}$. Then

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\tan(x))'}{(\sec(x))'}$$

Indeterminate limit $\dfrac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \dfrac{\sec(x)}{\tan(x)}$.

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The later limit is once again indeterminate, $\dfrac{\infty}{\infty}$. Then

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\tan(x))'}{(\sec(x))'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)}$$

Indeterminate limit $\dfrac{\infty}{\infty}$

Example

Evaluate $L = \displaystyle\lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\dfrac{\infty}{\infty}$. L'Hôpital's rule implies

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The later limit is once again indeterminate, $\dfrac{\infty}{\infty}$. Then

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\tan(x))'}{(\sec(x))'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}.$$

Indeterminate limit $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}$.

Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\sec(x))'}{(\tan(x))'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\tan(x)}{\sec(x)}.$$

The later limit is once again indeterminate, $\frac{\infty}{\infty}$. Then

$$L = \lim_{x \to (\frac{\pi}{2})^-} \frac{(\tan(x))'}{(\sec(x))'} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec^2(x)}{\sec(x)\tan(x)} = \lim_{x \to (\frac{\pi}{2})^-} \frac{\sec(x)}{\tan(x)}.$$

L'Hôpital's rule gives us a cycling expression. $\triangleleft$

Indeterminate limit $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to \infty} \frac{3x^2 - 5}{2x^2 - x + 3}$.

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Solution: This is an indeterminate limit $\frac{\infty}{\infty}$. L'Hôpital's rule implies

$$L = \lim_{x\to\infty} \frac{(3x^2 - 5)'}{(2x^2 - x + 3)'}$$

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Evaluate $L = \lim_{x \to \infty} \frac{3x^2 - 5}{2x^2 - x + 3}$.

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Recalling $\lim_{x\to\infty} \dfrac{1}{x} = 0$, we get that $L = \dfrac{6}{4}$. We conclude that

$$\lim_{x\to\infty} \frac{3x^2 - 5}{2x^2 - x + 3} = \frac{3}{2}. \qquad \triangleleft$$

Limits using L'Hôpital's Rule (Sect. 7.5)

- Review: L'Hôpital's rule for indeterminate limits $\dfrac{0}{0}$.
- Indeterminate limit $\dfrac{\infty}{\infty}$.
- **Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.**
- Overview of improper integrals (Sect. 8.7).

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

Remark: Sometimes limits of the form $\infty \cdot 0$ and $(\infty - \infty)$ can be converted by algebraic identities into indeterminate limits $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Example

Evaluate $L = \lim_{x \to 0} \Big(\frac{1}{\sin(x)} - \frac{1}{x} \Big)$.

Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.

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Solution: This is a limit of the form $(\infty - \infty)$. Since

$$\frac{1}{\sin(x)} - \frac{1}{x} = \frac{x - \sin(x)}{x \sin(x)}$$

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Then L'Hôpital's rule in this case implies

$$L = \lim_{x\to 0} \frac{\big(x - \sin(x)\big)'}{\big(x\sin(x)\big)'}$$

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$$L = \lim_{x \to 0} \frac{\big(x - \sin(x)\big)'}{\big(x \sin(x)\big)'} = \lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x) + x \cos(x)}$$

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Evaluate $L = \lim_{x \to 0} \left(\frac{1}{\sin(x)} - \frac{1}{x} \right)$.

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We conclude that $L = 0$. $\lhd$

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Evaluate $L = \lim_{x \to \infty} (3x)^{2/x}$.

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We conclude that $L = e^0$, that is, $L = 1$. $\triangleleft$

Limits using L'Hôpital's Rule (Sect. 7.5)

- Review: L'Hôpital's rule for indeterminate limits $\frac{0}{0}$.
- Indeterminate limit $\frac{\infty}{\infty}$.
- Indeterminate limits $\infty \cdot 0$ and $\infty - \infty$.
- **Overview of improper integrals (Sect. 8.7).**

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The improper integral of a continuous function $f : [a, \infty) \to \mathbb{R}$ is

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

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Overview of improper integrals (Sect. 8.7)

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Integrating by parts, $u = \ln(x)$, and $dv = dx/x^2$,

$$\int_1^{b} \frac{\ln(x)}{x^2}\, dx = \Big(-\frac{1}{x}\Big)\, \ln(x)\Big|_1^b - \int_1^b \Big(\frac{1}{x}\Big)\Big(-\frac{1}{x}\Big)\, dx$$

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Overview of improper integrals (Sect. 8.7)

Example

Evaluate $I = \int_1^{\infty} \frac{\ln(x)}{x^2}\, dx.$

Solution: Recall: $\displaystyle\int_1^{\infty} \frac{\ln(x)}{x^2}\, dx = \lim_{b\to\infty}\Big(-\frac{\ln(b)}{b} - \frac{1}{b} + 1\Big).$

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Improper integrals (Sect. 8.7)

This class:

- Integrals on infinite domains (Type I).
- The case $I = \displaystyle\int_1^{\infty} \frac{dx}{x^p}$.
- Integrands with vertical asymptotes (Type II).
- The case $I = \displaystyle\int_0^{1} \frac{dx}{x^p}$.

Next class:

- Convergence tests:
  - Direct comparison test.
  - Limit comparison test.
- Examples.

Improper integrals (Sect. 8.7)

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Therefore, the improper integral is

$$I = \lim_{b\to\infty} \Big[\ln\Big(\frac{b-2}{b}\Big) + \ln(3) \Big].$$

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We then conclude that $I = \ln(3)$. $\triangleleft$

Improper integrals (Sect. 8.7)

- Integrals on infinite domains (Type I).
- **The case $I = \int_1^\infty \frac{dx}{x^p}$.**
- Integrands with vertical asymptotes (Type II).
- The case $I = \int_0^1 \frac{dx}{x^p}$.

The case $I = \int_1^\infty \frac{dx}{x^p}$

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Evaluate $I = \int_1^\infty \frac{dx}{x^p}$ for $p > 0$.

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For $p \neq 1$, holds $I = \lim_{b \to \infty} \int_1^b x^{-p}\, dx$

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Improper integrals (Sect. 8.7)

- Integrals on infinite domains (Type I).
- The case $I = \int_1^\infty \frac{dx}{x^p}$.
- **Integrands with vertical asymptotes (Type II).**
- The case $I = \int_0^1 \frac{dx}{x^p}$.

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If f is continuous on $[a, c) \cup (c, b]$ and discontinuous at c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

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We conclude that I diverges. $\triangleleft$

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We conclude: $I = \frac{10}{3}$. ◁

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In the case $p = 1$ the integral diverges since $I = \lim_{a \to 0^+} \ln(a)$. $\triangleleft$

The cases $\int_0^1 \frac{dx}{x^p}$ and $\int_1^\infty \frac{dx}{x^p}$

Summary:

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} = \dfrac{1}{1-p} & p < 1, \\ \text{diverges} & p > 1. \end{cases}$$

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{diverges} & p < 1, \\ = \dfrac{1}{p-1} & p > 1. \end{cases}$$

In the case $p = 1$ both integrals diverge,

$$\int_0^1 \frac{dx}{x} = \text{diverges}, \qquad \int_1^\infty \frac{dx}{x} = \text{diverges}.$$