Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.
  - Solving differential equations (7.4).
  - Inverse trigonometric functions (7.6).
  - Hyperbolic functions (7.7).
  - Integration techniques (8-IT).
  - Integration by parts (8.1).
  - Trigonometric integrals (8.2).
- Section not covered:
  - Trigonometric substitutions (8.3).

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Solving differential equations (7.4)

Remark: Typical problems in this section:

(1) Find the function y solution of $y' = \dfrac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

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(2) The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

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We conclude that $y = -\sqrt{5 - \cos(x)}/\sqrt{2}$. $\triangleleft$

Solving differential equations (7.4)

Example

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$$L(x) = L_0\,e^{-kx}.$$

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so we conclude that $k = \ln(2)/15$.

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$$x_1 = \ln(8)\,\frac{15}{\ln(2)}$$

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$$x_1 = \ln(8)\,\frac{15}{\ln(2)} \quad\Rightarrow\quad x_1 = 3(15)$$

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Using the value $k = \ln(2)/15$, we get

$$x_1 = \ln(8)\,\frac{15}{\ln(2)} \quad\Rightarrow\quad x_1 = 3(15) \quad\Rightarrow\quad x_1 = 45. \qquad \triangleleft$$

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- **Inverse trigonometric functions (7.6).**
- Hyperbolic functions (7.7).
- Integration techniques (8-IT).
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- Trigonometric integrals (8.2).

Section not covered:

- Trigonometric substitutions (8.3).

Inverse trigonometric functions (7.6)

Notation: In the literature is common the notation $\sin^{-1} = \arcsin$, and similar for the rest of the trigonometric functions.

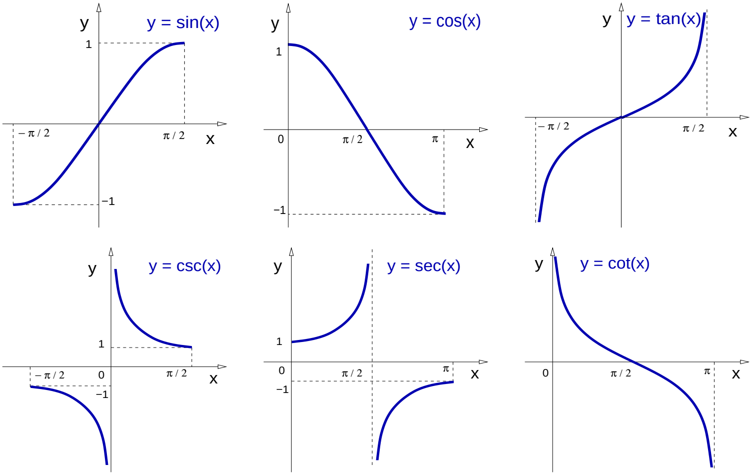
Do not confuse $\dfrac{1}{\sin(x)} \neq \sin^{-1}(x) = \arcsin(x)$.

Remark: sin, cos have simple values at particular angles.

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

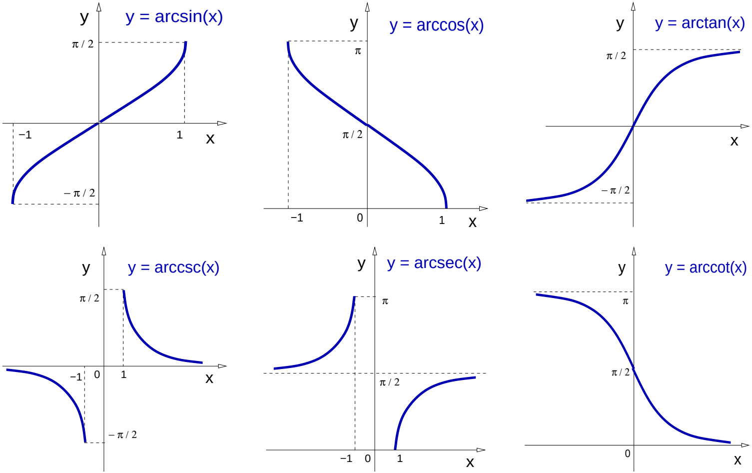
Inverse trigonometric functions (7.6)

Remark: On certain domains the trigonometric functions are invertible.



Inverse trigonometric functions (7.6)

Remark: The graph of the inverse function is a reflection of the original function graph about the $y = x$ axis.



Inverse trigonometric functions (7.6)

Theorem

The derivative of inverse trigonometric functions are:

$$\begin{aligned}
\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}}, & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}, & |x| \leqslant 1,\\
\arctan'(x) &= \frac{1}{1+x^2}, & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2}, & x \in \mathbb{R},\\
\operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}}, & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}, & |x| \geqslant 1.
\end{aligned}$$

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Recall $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}$,

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$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| \geqslant 1.$$

Recall $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}$, $\tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

Inverse trigonometric functions (7.6)

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Inverse trigonometric functions (7.6)

Remark: Typical problems in this section:

(1) Sketch the graphs of

$$y(x) = \sec(x), \qquad z(x) = \operatorname{arcsec}(x).$$

State the respective domains and ranges.

(2) Evaluate $\cos\big(\arcsin(1/\sqrt(2))\big)$.

(3) Evaluate $\sec\big(\arctan(-2/3)\big)$.

(4) Find y' for $y(x) = \arctan(3x^2)$.

(5) Find $I = \displaystyle\int \frac{dx}{\sqrt{2 - x^2}}$.

Inverse trigonometric functions (7.6)

Example

Evaluate $\sec\big(\arctan(-2/3)\big)$.

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Inverse trigonometric functions (7.6)

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Inverse trigonometric functions (7.6)

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Then holds $\sec(\theta) = \pm\sqrt{\tan^2(\theta) + 1}$. We need to find the correct sign: $\theta = \arctan(-2/3) \in (-\pi/2, 0)$. Since $\sec(\theta) = 1/\cos(\theta)$,

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Inverse trigonometric functions (7.6)

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Evaluate $\sec\big(\arctan(-2/3)\big)$.

Solution: We only need the relation between sec and tan,

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$$\sec(\arctan\Big(-\frac{2}{3}\Big)) = \sqrt{\tan^2(\arctan\Big(-\frac{2}{3}\Big)) + 1}$$

Inverse trigonometric functions (7.6)

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Solution: We only need the relation between sec and tan,

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$$\sec(\arctan\Big(-\frac{2}{3}\Big)) = \sqrt{\tan^2(\arctan\Big(-\frac{2}{3}\Big)) + 1} = \sqrt{\frac{4}{9} + 1}$$

Inverse trigonometric functions (7.6)

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Solution: We only need the relation between sec and tan,

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$$\sec(\arctan\Bigl(-\frac{2}{3}\Bigr)) = \sqrt{\tan^2(\arctan\Bigl(-\frac{2}{3}\Bigr)) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.$$

Inverse trigonometric functions (7.6)

Example

Evaluate $\sec\big(\arctan(-2/3)\big)$.

Solution: We only need the relation between sec and tan,

$$\sec^2(\theta) = \tan^2(\theta) + 1.$$

Then holds $\sec(\theta) = \pm\sqrt{\tan^2(\theta) + 1}$. We need to find the correct sign: $\theta = \arctan(-2/3) \in (-\pi/2, 0)$. Since $\sec(\theta) = 1/\cos(\theta)$, we conclude that $\sec(\theta) > 0$. Hence

$$\sec(\arctan\Big(-\frac{2}{3}\Big)) = \sqrt{\tan^2(\arctan\Big(-\frac{2}{3}\Big)) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.$$

We conclude that $\sec(\arctan(-2/3)) = \sqrt{13}/3$. $\triangleleft$

Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.

- Solving differential equations (7.4).
- Inverse trigonometric functions (7.6).
- **Hyperbolic functions (7.7).**
- Integration techniques (8-IT).
- Integration by parts (8.1).
- Trigonometric integrals (8.2).

Section not covered:

- Trigonometric substitutions (8.3).

Hyperbolic functions (7.7)

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Hyperbolic functions (7.7)

Remark: Typical problems in this section:

(1) Prove the identities: $\cosh^2(x) - \sinh^2(x) = 1$, and

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad \sinh(2x) = 2\sinh(x)\cosh(x)),$$

$$\cosh^2(x) = \frac{1}{2}\big(1 + \cosh(2x)\big), \qquad \sinh^2(x) = \frac{1}{2}\big(-1 + \cosh(2x)\big).$$

(2) Know the derivatives and integrals of hyperbolic functions.

Review for Exam 2.

Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.

- Solving differential equations (7.4).
- Inverse trigonometric functions (7.6).
- Hyperbolic functions (7.7).
- **Integration techniques (8-IT).**
- **Integration by parts (8.1).**
- **Trigonometric integrals (8.2).**

Section not covered:

- Trigonometric substitutions (8.3).

Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) $\displaystyle\int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

(2) $\displaystyle\int_1^8 \frac{dx}{x^2-2x+50}$.

(3) $\displaystyle\int x^3 \ln(x)\,dx$.

(4) $\displaystyle\int x^2\, e^{2x}\,dx$.

(5) $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$.

(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$, $|x|<5$.

(7) $\displaystyle\int \cot^3(x)\,dx$.

(8) $\displaystyle\int \sin^4(x)\,dx$.

(9) $\displaystyle\int x^3 \cos(x)\,dx$.

(10) $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos(2x)}\,dx$.

(11) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)}\,dx$.

(12) $\displaystyle\int \frac{2^{\ln(x)}}{x}\,dx$.

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

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Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) $\displaystyle\int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$. Split the integral and do two substitutions.

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$, $|x| < 5$.

Sections 8-IT, 8.1, 8.2

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(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$, $|x| < 5$. Substitution

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(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$, $|x|<5$. Substitution and recall $\arcsin'$.

Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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(7) $\int \cot^3(x)\,dx$. Write using sin, cos and substitution.

(8) $\int \sin^4(x)\,dx$. Double angle formula,

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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(9) $\displaystyle\int x^3\cos(x)\,dx$. Integrations by parts, three times.

(10) $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos(2x)}\,dx$.

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

Solution: Split the integral: $I = \int \frac{dx}{\sqrt{1-2x^2}} + \int \frac{x\,dx}{\sqrt{1-2x^2}}$.

Example

Evaluate $I = \int \dfrac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

Solution: Split the integral: $I = \int \dfrac{dx}{\sqrt{1-2x^2}} + \int \dfrac{x\,dx}{\sqrt{1-2x^2}}$.

For the first integral

Sections 8-IT, 8.1, 8.2

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For the second integral

Sections 8-IT, 8.1, 8.2

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For the second integral substitute $u = 1 - 2x^2$, then $du = -4x\,dx$.

$$I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}}$$

Sections 8-IT, 8.1, 8.2

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For the second integral substitute $u = 1 - 2x^2$, then $du = -4x\,dx$.

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Sections 8-IT, 8.1, 8.2

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We conclude: $I = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}\,x) - \frac{1}{2}\sqrt{1-2x^2} + c.$ $\triangleleft$

Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int \frac{dx}{\sqrt{8x - x^2}}$.

Sections 8-IT, 8.1, 8.2

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Evaluate $I = \int \frac{dx}{\sqrt{8x - x^2}}$.

Solution: Complete the square and recall $\arcsin'$.

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)}\, dx.$

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Recall: $\sin^2(\theta) = [1 - \cos(2\theta)]/2.$

Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Sections 8-IT, 8.1, 8.2

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Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- Polynomial division: $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}, \quad k < n.$
- The method of partial fractions.
  - The case $\dfrac{p_1(x)}{(x - r_1)(x - r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
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Integrating rational functions

Remark:

The problem is to integrate rational functions $f(x) = \frac{p_m(x)}{q_n(x)}$, where $p_m(x)$, $q_m(x)$ are polynomials degree m, and n.

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It can be proven that $\frac{(5x-3)}{(x^2-2x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$.

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Remark: We now present a method to simplify functions $f(x) = \dfrac{p_m(x)}{q_n(x)}$, as additions of functions simpler to integrate.

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Verify that $\dfrac{4x^2 - 7}{2x + 3} = 2x - 3 + \dfrac{2}{2x + 3}$.

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Solution:

$$2x - 3 + \frac{2}{2x+3} = \frac{(2x-3)(2x+3)+2}{2x+3} = \frac{4x^2 - 9 + 2}{2x+3}. \quad \triangleleft$$

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$$
\begin{array}{r}
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\end{array}
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 & \quad\quad\quad\quad 2x - 3 \\
\hline
2x + 3 & \;\; 4x^2 \quad\quad\quad - 7 \\
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\cline{2-2}
 & \quad\quad -6x - 7 \\
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$$\Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.$$

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$$I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3}$$

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\hline
 & \qquad\qquad 2
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$$I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3} \qquad \Rightarrow \qquad I = x^2 - 3x + \ln(2x + 3) + c.$$

Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- Polynomial division: $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}, \quad k < n.$
- **The method of partial fractions.**
  - The case $\dfrac{p_1(x)}{(x-r_1)(x-r_2)} \quad r_1 \neq r_2$ (Non-repeated roots).
  - The case $\dfrac{p_{(n-1)}(x)}{(x-r_1)^n}$. (Repeated roots).
  - The case $\dfrac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}, \quad b^2 - 4c < 0$ (Complex roots).
  - The general case.

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Remarks:

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- We present the method through examples.
- We go from simpler to more complicated situations.

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The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{1}{(x-1)(x+2)}\, dx.$

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Solution: Denote $r_1 = 1$, $r_2 = -2$. Find a_1 and a_2 such that

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$$I = \int \frac{1}{(x-1)(x-2)}\,dx = \int \frac{1}{3}\,\frac{1}{(x-1)}\,dx - \int \frac{1}{3}\,\frac{1}{(x+2)}\,dx$$

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We conclude that

$$I = \frac{1}{3}\,\ln|x-1| - \frac{1}{3}\,\ln|x+2| + c. \qquad \triangleleft$$

The method of partial fractions (Non-repeated roots)

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Evaluate $I = \int \frac{(x-1)}{(x^2-x-2)}\, dx.$

The method of partial fractions (Non-repeated roots)

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Evaluate $I = \int \frac{(x-1)}{(x^2 - x - 2)}\, dx.$

Solution: First, find the zeros of the denominator,

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Evaluate $I = \int \frac{(x-1)}{(x^2-x-2)}\, dx.$

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Solution: First, find the zeros of the denominator,

$$x^2 - x - 2 = 0 \quad \Rightarrow \quad x_{\pm} = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big]$$

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Therefore, we rewrite: $I = \int \frac{(x-1)}{(x-2)(x+1)}\, dx.$

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Partial fraction problem: Find constants a_1 and a_2 such that

$$\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1}{(x-2)} + \frac{a_2}{(x+1)}, \quad r_1 = 2, \quad r_2 = -1.$$

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Do the addition on the right-hand side above:

$$\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1(x+1) + a_2(x-2)}{(x-2)(x+1)}.$$

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Evaluate $I = \int \frac{(x-1)}{(x^2 - x - 2)}\, dx.$

Solution: Recall: $\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1(x+1) + a_2(x-2)}{(x-2)(x+1)}.$

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The equation above implies:

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We obtain $\dfrac{(x-1)}{(x-2)(x+1)} = \dfrac{1}{3}\,\dfrac{1}{(x-2)} + \dfrac{2}{3}\,\dfrac{1}{(x+1)}.$

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The method of partial fractions (Non-repeated roots)

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The method of partial fractions (Non-repeated roots)

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We conclude that

$$I = \frac{1}{3}\,\ln|x-2| + \frac{2}{3}\,\ln|x+1| + c.$$

$\triangleleft$

The method of partial fractions (Non-repeated roots)

Theorem (Non-repeated roots - Heaviside cover method)

The rational function $\dfrac{p_k(x)}{(x-r_1)\cdots(x-r_n)}$*, with* $n > k$ *and all roots* $r_1, \cdots, r_n$ *different, can be written as*

$$\frac{p_k(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{(x-r_1)} + \cdots + \frac{a_n}{(x-r_n)},$$

where the constants $a_1, \cdots, a_n$ *are given by*

$$a_1 = \frac{p_k(r_1)}{\prod_{j\neq 1}(r_1 - r_j)}, \quad \cdots \quad a_n = \frac{p_k(r_n)}{\prod_{j\neq n}(r_n - r_j)}.$$

The method of partial fractions (Non-repeated roots)

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Proof: $p_k(x) = a_1 \left[\prod_{j\neq 1}(x - r_j)\right] + \cdots + a_n \left[\prod_{j\neq n}(x - r_j)\right].$ □

Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
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  - The case $\dfrac{p_1(x)}{(x-r_1)(x-r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
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  - The case $\dfrac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}$, $b^2 - 4c < 0$ (Complex roots).
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The method of partial fractions (Repeated roots)

Example

Evaluate $I = \int \frac{(2x-1)}{(x^2-6x+9)}\, dx.$

The method of partial fractions (Repeated roots)

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The method of partial fractions (Repeated roots)

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The method of partial fractions (Repeated roots)

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Partial fraction problem: Find constants a_1 and a_2 such that

$$\frac{(2x-1)}{(x-3)^2} = \frac{a_1}{(x-3)} + \frac{a_2}{(x-3)^2}.$$

The method of partial fractions (Repeated roots)

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Do the addition on the right-hand side above:

$$\frac{(2x-1)}{(x-3)^2} = \frac{a_1(x-3)+a_2}{(x-3)^2}.$$

The method of partial fractions (Repeated roots)

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Evaluate $I = \int \frac{(2x-1)}{(x^2-6x+9)}\,dx.$

Solution: Recall: $\frac{(2x-1)}{(x-3)^2} = \frac{a_1(x-3)+a_2}{(x-3)^2}.$

The method of partial fractions (Repeated roots)

Example

Evaluate $I = \int \frac{(2x-1)}{(x^2 - 6x + 9)}\, dx.$

Solution: Recall: $\frac{(2x-1)}{(x-3)^2} = \frac{a_1(x-3) + a_2}{(x-3)^2}$. Then,

$$2x - 1 = a_1(x-3) + a_2.$$

The method of partial fractions (Repeated roots)

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To compute a_2 evaluate the expression above at $r=3$,

$$5 = a_2.$$

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To compute a_1 derivate the expression above, then evaluate at $r = 3$, (the evaluation at $r = 3$ is not needed in this case),

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We conclude: $\frac{(2x-1)}{(x-3)^2} = \frac{2}{(x-3)} + \frac{5}{(x-3)^2}.$

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The method of partial fractions (Repeated roots)

Theorem (Repeated roots)

The rational function $\dfrac{p_k(x)}{(x-r)^n}$*, with* $n > k$*, can be written as*

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Proof: Taking common denominator on the right-hand side above,

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Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
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  - **The case** $\dfrac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}$**,** $b^2 - 4c < 0$ **(Complex roots).**
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The method of partial fractions (Complex roots)

Example

Evaluate $I = \int \frac{(x+1)^2}{(x^2+1)^2}\,dx.$

The method of partial fractions (Complex roots)

Example

Evaluate $I = \int \frac{(x+1)^2}{(x^2+1)^2}\, dx$.

Solution: Find constants a_1, b_1 and a_2, b_2 such that

$$\frac{(x+1)^2}{(x^2+1)^2} = \frac{(a_1 x + b_1)}{(x^2+1)} + \frac{(a_2 x + b_2)}{(x^2+1)^2}.$$

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$$x^2 + 2x + 1 = a_1 x^3 + a_1 x + b_1 x^2 + b_1 + a_2 x + b_2.$$

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$$x^2+2x+1 = a_1x^3+a_1x+b_1x^2+b_1+a_2x+b_2.$$

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Evaluate $I = \int \frac{(x+1)^2}{(x^2+1)^2}\, dx.$

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We conclude: $a_1 = 0$,

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We conclude: $a_1 = 0$, $b_1 = 1$, $a_2 = 2$, and $b_2 = 0$. Hence,

$$I = \int \frac{(a_1 x + b_1)}{(x^2+1)}\,dx + \int \frac{(a_2 x + b_2)}{(x^2+1)^2}\,dx.$$

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$$I = \int \frac{dx}{x^2+1} + \int \frac{2x\,dx}{(x^2+1)^2}.$$

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We conclude that $I = \arctan(x) - \frac{1}{(x^2+1)} + c.$ ◁

The method of partial fractions (Complex roots)

Theorem (Repeated roots)

The rational function $\dfrac{p_{(2n-1)}(x)}{(x+bx+c)^n}$, *with* $b^2-4c<0$, *can be written as*

$$\frac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}=\frac{a_1x+b_1}{(x^2+bx+c)}+\cdots+\frac{a_nx+b_n}{(x^2+bx+c)^n}$$

for appropriate constants a_i, b_1 *for* $i=1,\cdots,n$.

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for appropriate constants a_i, b_1 *for* $i = 1, \cdots, n$.

Idea of the Proof:

Taking common denominator on the right-hand side above,

$$p_{(2n-1)}(x) = (a_1x+b_1)(x^2+bx+c)^{(n-1)} + \cdots + (a_nx+b_n).$$

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Theorem (Repeated roots)

The rational function $\dfrac{p_{(2n-1)}(x)}{(x+bx+c)^n}$*, with* $b^2-4c<0$*, can be written as*

$$\frac{p_{(2n-1)}(x)}{(x^2+bx+c)^n} = \frac{a_1 x + b_1}{(x^2+bx+c)} + \cdots + \frac{a_n x + b_n}{(x^2+bx+c)^n}$$

for appropriate constants a_i*,* b_1 *for* $i = 1, \cdots, n$*.*

Idea of the Proof:

Taking common denominator on the right-hand side above,

$$p_{(2n-1)}(x) = (a_1 x + b_1)(x^2+bx+c)^{(n-1)} + \cdots + (a_n x + b_n).$$

Expanding the equation above one can find a system of equations for the coefficients.

Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- Polynomial division: $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}, \quad k < n.$
- The method of partial fractions.
  - The case $\dfrac{p_1(x)}{(x - r_1)(x - r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
  - The case $\dfrac{p_{(n-1)}(x)}{(x - r_1)^n}$. (Repeated roots).
  - The case $\dfrac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}$, $b^2 - 4c < 0$ (Complex roots).
  - **The general case.**

The method of partial fractions (General case)

Remarks:

- Consider a general rational function $\dfrac{r_k(x)}{q_n(x)}$, with $k < n$.

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- Express the denominator, q, as a product of factors $(x - r_i)^{m_i}$ and $(x^2 + b_i x + c_i)^{\ell_i}$, with r_i roots of q_n, and $b_i^2 - 4c_i < 0$.

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- The partial fraction decomposition for $\dfrac{r_k}{q_n}$ is the addition of the partial fraction decomposition for each factor in q.

The method of partial fractions (General case)

Example

Evaluate $I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x}\, dx.$

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$$a+c+d = 6, \quad -2a+b-2d = -8, \quad 5 = -2b+c+d \quad d = 3.$$

The method of partial fractions (General case)

Example

Evaluate $I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x}\,dx.$

Solution: Recall:

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$$I = \frac{1}{2}\ln(x^2+1) + 2\ln|x-2| + 3\ln|x| + c. \qquad \triangleleft$$