Integration by parts (Sect. 8.1)

- Integral form of the product rule.
- Exponential and logarithms.
- Trigonometric functions.
- Definite integrals.
- Substitution and integration by parts.

Integral form of the product rule

Remark: The integration by parts formula is an integral form of the product rule for derivatives:

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Theorem

For all differentiable functions $g, f : \mathbb{R} \to \mathbb{R}$ *holds*

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx.$$

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Notation: It is common to write $\int u\, dv = uv - \int v\, du$, where

$$u = f(x), \quad dv = g'(x)\, dx, \quad \text{and} \quad v = g(x), \quad du = f'(x)\, dx.$$

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Exponentials and logarithms

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Evaluate $I = \int x\, e^{2x}\, dx$.

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We need to choose the functions f and g.

Exponentials and logarithms

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Solution: Recall: $\displaystyle \int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx.$

We need to choose the functions f and g.

They should be chosen in a way that the right-hand side above is simpler to integrate than the original left-hand side.

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$$f(x) = x, \quad g'(x) = e^{2x}$$

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$$f(x) = x, \quad g'(x) = e^{2x} \quad \Rightarrow \quad f'(x) = 1, \quad g(x) = \frac{e^{2x}}{2}.$$

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We conclude $I = \frac{e^{2x}}{4}(2x - 1) + c$. $\triangleleft$

Exponentials and logarithms

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$$f(x) = e^{2x}, \quad g'(x) = x \quad \Rightarrow \quad f'(x) = 2\, e^{2x}, \quad g(x) = \frac{x^2}{2}.$$

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Example

Evaluate $I = \int \ln(x)\,dx$.

Solution: We need to choose u and v:

$$u = \ln(x), \quad dv = dx \quad \Rightarrow \quad du = \frac{dx}{x}, \quad v = x.$$

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Evaluate $I = \int \ln(x)\,dx$.

Solution: We need to choose u and v:

$$u = \ln(x), \quad dv = dx \quad \Rightarrow \quad du = \frac{dx}{x}, \quad v = x.$$

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We conclude that $I = x\,\ln(x) - x + c$. $\triangleleft$

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Solution: $u = x^2, \quad dv = e^x \, dx \quad \Rightarrow \quad du = 2x \, dx, \quad v = e^x.$

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Evaluate $I = \int x^2\, e^x\, dx$.

Solution: $u = x^2, \quad dv = e^x\, dx \quad \Rightarrow \quad du = 2x\, dx, \quad v = e^x$.

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We integrate by parts one more time.

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We conclude that $I = x^2\, e^x - 2x\, e^x + 2e^x + c.$ $\lhd$

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Evaluate $I = \int x \sin(x)\, dx.$

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We conclude that $I = -x\cos(x) + \sin(x) + c.$ $\lhd$

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Evaluate $I = \displaystyle\int e^{ax} \sin(x)\, dx.$

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$$u = e^{ax}, \quad dv = \cos(x)\, dx \quad \Rightarrow \quad du = a\, e^{ax}\, dx, \quad v = \sin(x).$$

$$I = -e^{ax}\cos(x) + a\Big[e^{ax}\sin(x) - a \int e^{ax}\sin(x)\, dx\Big].$$

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Remark: The last term on the right-hand side is proportional to the negative of the left-hand side.

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Remark: The last term on the right-hand side is proportional to the negative of the left-hand side. So, for all $a \neq 0$ holds

$$(1 + a^2) \int e^{ax} \sin(x)\, dx = -e^{ax} \cos(x) + a\, e^{ax} \sin(x).$$

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$$(1 + a^2) \int e^{ax} \sin(x)\, dx = -e^{ax} \cos(x) + a\, e^{ax} \sin(x).$$

We then conclude that

$$\int e^{ax} \sin(x)\, dx = \frac{e^{ax}}{(1 + a^2)} \big[- \cos(x) + a \sin(x) \big]. \qquad \triangleleft$$

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Remark: Integration by parts can be used with definite integrals.

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Theorem

For all differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ *holds*

$$\int_a^b f(x)\, g'(x)\, dx = \big[f(x)\, g(x)\big]\Big|_a^b - \int_a^b f'(x)\, g(x)\, dx.$$

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Evaluate $I = \displaystyle\int_0^{\pi} e^{ax} \sin(x)\, dx.$

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Example

Evaluate $\quad I = \displaystyle\int_0^\pi e^{ax} \sin(x)\, dx.$

Solution: Use integrations by parts and evaluate the result:

$$\int_0^\pi e^{ax} \sin(x)\, dx = \left[\frac{e^{ax}}{(1+a^2)} \big[-\cos(x) + a \sin(x) \big] \right] \Bigg|_0^\pi$$

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Example

Evaluate $I = \int_0^{\pi} e^{ax} \sin(x)\, dx.$

Solution: Use integrations by parts and evaluate the result:

$$\int_0^{\pi} e^{ax} \sin(x)\, dx = \Big[\frac{e^{ax}}{(1+a^2)} \big[-\cos(x) + a \sin(x) \big] \Big]\Big|_0^{\pi}$$

$$\int_0^{\pi} e^{ax} \sin(x)\, dx = \frac{(e^{a\pi}+1)}{(1+a^2)}. \qquad \triangleleft$$

Integration by parts (Sect. 8.1)

- Integral form of the product rule.
- Exponential and logarithms.
- Trigonometric functions.
- Definite integrals.
- **Substitution and integration by parts.**

Substitution and integration by parts

Remark: Substitution and integration by parts can be used on the same integral.

Example

Evaluate $I = \int \cos\big(\ln(x)\big)\, dx.$

Substitution and integration by parts

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Evaluate $I = \int \cos\big(\ln(x)\big)\,dx$.

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Substitution and integration by parts

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Example

Evaluate $I = \int \cos(\ln(x))\, dx$.

Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

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Example

Evaluate $I = \int \cos\big(\ln(x)\big)\, dx$.

Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

$$dx = x\, dy$$

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Example

Evaluate $I = \int \cos\big(\ln(x)\big)\, dx$.

Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

$$dx = x\, dy = e^y\, dy.$$

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Evaluate $I = \int \cos\big(\ln(x)\big)\,dx$.

Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

$$dx = x\,dy = e^y\,dy.$$

The integral is $I = \int \cos(y)\,e^y\,dy$,

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Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

$$dx = x\, dy = e^y\, dy.$$

The integral is $I = \int \cos(y)\, e^y\, dy$, and we integrate by parts.

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The integral is $I = \int \cos(y)\, e^y\, dy$, and we integrate by parts.

If $u = e^y$, $dv = \cos(y)\, dy$, then $du = e^y\, dy$, $v = \sin(y)$,

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If $u = e^y$, $dv = \cos(y)\, dy$, then $du = e^y\, dy$, $v = \sin(y)$,

$$\int e^y \cos(y)\, dy = e^y \sin(y) - \int e^y \sin(y)\, dy.$$

Substitution and integration by parts

Example

Evaluate $I = \int \cos(\ln(x))\,dx$.

Solution: Recall: $\int e^y \cos(y)\,dy = e^y \sin(y) - \int e^y \sin(y)\,dy.$

Substitution and integration by parts

Example

Evaluate $I = \int \cos\big(\ln(x)\big)\, dx$.

Solution: Recall: $\int e^y \cos(y)\, dy = e^y \sin(y) - \int e^y \sin(y)\, dy$.

One more integration by parts,

Substitution and integration by parts

Example

Evaluate $I = \int \cos\big(\ln(x)\big)\,dx.$

Solution: Recall: $\displaystyle\int e^y \cos(y)\,dy = e^y\,\sin(y) - \int e^y\,\sin(y)\,dy.$

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$$u = e^y, \quad dv = \sin(y)\,dy, \quad \Rightarrow \quad du = e^y\,dy, \quad v = -\cos(y).$$

Substitution and integration by parts

Example

Evaluate $I = \int \cos\big(\ln(x)\big)\, dx.$

Solution: Recall: $\int e^y \cos(y)\, dy = e^y \sin(y) - \int e^y \sin(y)\, dy.$

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$$\int e^y \cos(y)\, dy = e^y \sin(y) - \Big[-e^y \cos(y) + \int e^y \cos(y)\, dy\Big].$$

Substitution and integration by parts

Example

Evaluate $I = \int \cos(\ln(x))\,dx$.

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$$2\int e^y \cos(y)\,dy = e^y\,\sin(y) + e^y\,\cos(y)$$

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$$2\int e^y \cos(y)\, dy = e^y\, \sin(y) + e^y\, \cos(y)$$

$$\int e^y \cos(y) = \frac{e^y}{2}\big[\sin(y) + \cos(y)\big].$$

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$$u = e^y, \quad dv = \sin(y)\,dy, \quad \Rightarrow \quad du = e^y\,dy, \quad v = -\cos(y).$$

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$$2\int e^y \cos(y)\,dy = e^y\,\sin(y) + e^y\,\cos(y)$$

$$\int e^y \cos(y) = \frac{e^y}{2}\big[\sin(y) + \cos(y)\big].$$

We conclude: $\displaystyle\int \cos\big(\ln(x)\big)\,dx = \frac{x}{2}\big[\sin(\ln(x)) + \cos(\ln(x))\big].$ $\triangleleft$

Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- Integrals of tangents and secants.
- Products of sines and cosines.

Product of sines and cosines

Remark: There is a procedure to compute integrals of the form

$$I = \int \sin^m(x)\,\cos^n(x)\,dx.$$

Product of sines and cosines

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$$I = \int \sin^m(x) \cos^n(x)\, dx.$$

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Product of sines and cosines

Remark: There is a procedure to compute integrals of the form

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Remark: There is a procedure to compute integrals of the form

$$I = \int \sin^m(x)\cos^n(x)\,dx.$$

(a) If $m = 2k + 1$, (odd), then $\sin^{(2k+1)}(x) = \big(\sin^2(x)\big)^k \sin(x)$;

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(a) If $m = 2k+1$, (odd), then $\sin^{(2k+1)}(x) = \big(\sin^2(x)\big)^k\,\sin(x)$;

$$I = \int \big(1 - \cos^2(x)\big)^k\,\cos^n(x)\,\sin(x)\,dx.$$

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Substitute $u = \cos(x)$, so $du = -\sin(x)\, dx$,

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Substitute $u = \cos(x)$, so $du = -\sin(x)\,dx$, hence

$$I = -\int (1-u^2)^k\,u^n\,du.$$

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Substitute $u = \cos(x)$, so $du = -\sin(x)\, dx$, hence

$$I = -\int (1 - u^2)^k\, u^n\, du.$$

We now need to integrate a polynomial.

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$$I = \int \sin^m(x)\,\big(1 - \sin^2(x)\big)^k \cos(x)\,dx.$$

Substitute $u = \sin(x)$, so $du = \cos(x)\,dx$, hence

$$I = \int u^m\,(1 - u^2)^k\,du.$$

Again, we now need to integrate a polynomial.

Product of sines and cosines

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$$I = \int \sin^m(x) \cos^n(x)\, dx.$$

(c) If both m and n are even,

Product of sines and cosines

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Now use the identities

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right),$$

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Now use the identities

$$\sin^2(x) = \frac{1}{2}\left(1 - \cos(2x)\right), \qquad \cos^2(x) = \frac{1}{2}\left(1 + \cos(2x)\right).$$

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Depending whether k or ℓ are odd,

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Depending whether k or ℓ are odd, repeat (a), (b) or (c).

Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\, dx.$

Product of sines and cosines

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Solution: Since $m = 5$ is odd,

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Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\, dx$.

Solution: Since $m = 5$ is odd, we write it as $m = 4 + 1$,

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Product of sines and cosines

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Evaluate $I = \int \sin^5(x)\,dx.$

Solution: Since $m = 5$ is odd, we write it as $m = 4 + 1$,

$$I = \int \sin^{4+1}(x)\,dx = \int \big(\sin^2(x)\big)^2 \sin(x)\,dx$$

$$I = \int \big(1 - \cos^2(x)\big)^2 \sin(x)\,dx.$$

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Introduce the substitution

Product of sines and cosines

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Introduce the substitution $u = \cos(x)$,

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Introduce the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$,

$$I = -\int (1 - u^2)^2\,du$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\,dx$.

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$$I = \int \big(1 - \cos^2(x)\big)^2 \sin(x)\,dx.$$

Introduce the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$,

$$I = -\int (1-u^2)^2\,du = -\int (1 - 2u^2 + u^4)\,du.$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\,dx$.

Solution: Since $m = 5$ is odd, we write it as $m = 4 + 1$,

$$I = \int \sin^{4+1}(x)\,dx = \int \big(\sin^2(x)\big)^2\,\sin(x)\,dx$$

$$I = \int \big(1 - \cos^2(x)\big)^2\,\sin(x)\,dx.$$

Introduce the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$,

$$I = -\int (1 - u^2)^2\,du = -\int (1 - 2u^2 + u^4)\,du.$$

$$I = -u + 2\,\frac{u^3}{3} - \frac{u^5}{5} + c.$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^5(x)\,dx$.

Solution: Since $m = 5$ is odd, we write it as $m = 4 + 1$,

$$I = \int \sin^{4+1}(x)\,dx = \int \big(\sin^2(x)\big)^2\,\sin(x)\,dx$$

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Introduce the substitution $u = \cos(x)$, then $du = -\sin(x)\,dx$,

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$$I = -u + 2\,\frac{u^3}{3} - \frac{u^5}{5} + c.$$

We conclude $I = -\cos(x) + \dfrac{2}{3}\cos^3(x) - \dfrac{1}{5}\,\cos^5(x) + c.$ $\triangleleft$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\, dx.$

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Example

Evaluate $I = \int \sin^6(x)\,dx.$

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Evaluate $I = \int \sin^6(x)\, dx.$

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

Product of sines and cosines

Example

Evaluate $I = \displaystyle\int \sin^6(x)\,dx.$

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \big(\sin^2(x)\big)^3\,dx$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \left(\sin^2(x)\right)^3 dx = \int \left(\frac{1}{2}[1 - \cos(2x)]\right)^3 dx$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\, dx.$

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \left(\sin^2(x)\right)^3 dx = \int \left(\frac{1}{2}[1 - \cos(2x)]\right)^3 dx$$

$$I = \frac{1}{8} \int \left(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\right) dx.$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \big(\sin^2(x)\big)^3\,dx = \int \Big(\frac{1}{2}[1 - \cos(2x)]\Big)^3\,dx$$

$$I = \frac{1}{8}\int \big(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\big)\,dx.$$

The first two terms are: $\int (1 - 3\cos(2x))\,dx = x - \frac{3}{2}\sin(2x)$.

Product of sines and cosines

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$$I = \int \big(\sin^2(x)\big)^3\, dx = \int \Big(\frac{1}{2}[1 - \cos(2x)]\Big)^3\, dx$$

$$I = \frac{1}{8}\int \big(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\big)\, dx.$$

The first two terms are: $\int (1 - 3\cos(2x))\, dx = x - \frac{3}{2}\sin(2x)$.

The third term can be integrated as follows,

$$\int 3\cos^2(2x)\, dx = 3\int \frac{1}{2}(1 + \cos(4x))\, dx$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

Solution: Since $m = 6$ is even, we write it as $m = 2(3)$,

$$I = \int \left(\sin^2(x)\right)^3 dx = \int \left(\frac{1}{2}[1 - \cos(2x)]\right)^3 dx$$

$$I = \frac{1}{8}\int \left(1 - 3\cos(2x) + 3\cos^2(2x) - \cos^3(2x)\right) dx.$$

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Product of sines and cosines

Example

Evaluate $I = \displaystyle\int \sin^6(x)\, dx.$

Solution: So far we have found that

$$I = \frac{1}{8}\left[x - \frac{3}{2}\sin(2x) + \frac{3}{2}\left(x + \frac{1}{4}\sin(4x)\right)\right] - \frac{1}{8}\int \cos^3(2x)\, dx.$$

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Product of sines and cosines

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Introduce the substitution $u = \sin(2x)$, then $du = 2\cos(2x)\, dx$.

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$$J = \frac{1}{2}\int (1 - u^2)\, du$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

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Introduce the substitution $u = \sin(2x)$, then $du = 2\cos(2x)\,dx$.

$$J = \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\Big(u - \frac{u^3}{3}\Big)$$

Product of sines and cosines

Example

Evaluate $I = \int \sin^6(x)\,dx$.

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$$I = \frac{1}{8}\Big[x - \frac{3}{2}\,\sin(2x) + \frac{3}{2}x + \frac{3}{8}\sin(4x) - \frac{1}{2}\sin(2x) + \frac{1}{6}\sin^3(2x)\Big] + c.$$

Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- **Eliminating square roots.**
- Integrals of tangents and secants.
- Products of sines and cosines.

Eliminating square roots

Remarks:

- Recall the double angle identities:

$$\sin^2(\theta) = \frac{1}{2}\big[1 - \cos(2\theta)\big], \qquad \cos^2(\theta) = \frac{1}{2}\big[1 + \cos(2\theta)\big].$$

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Example

Evaluate $I = \int_0^{\pi/8} \sqrt{1 + \cos(8x)}\, dx$.

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Example

Evaluate $I = \int_0^{\pi/8} \sqrt{1 + \cos(8x)}\, dx$.

Solution: Use that : $1 + \cos(8x) = 2\cos^2(4x)$. Hence,

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Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- **Integrals of tangents and secants.**
- Products of sines and cosines.

Integrals of tangents and secants

Remark: Recall the identities:

$$\tan'(x) = \sec^2(x) = \tan^2(x) + 1.$$

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First equation comes from quotient rule, the second from Pythagoras Theorem. These identities can be used to compute

$$I = \int \tan^{2k}(x)\,dx, \qquad k \in \mathbb{N}.$$

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Evaluate $I = \displaystyle\int \tan^2(x)\,dx$.

Solution: The identity on the far left above implies

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Example

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Solution: The identity on the far left above implies

$$I = \int \big(\tan'(x) - 1\big)\,dx \quad \Rightarrow \quad I = \tan(x) - x + c. \qquad \triangleleft$$

Integrals of tangents and secants

Example

Find a recurrence formula to compute $I = \int \tan^{2k}(x)\, dx$, $k \in \mathbb{N}$.

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Integrals of tangents and secants

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$$I = \frac{1}{(2k-1)}\, \tan^{(2k-1)}(x) - \int \tan^{2(k-1)}(x)\,dx. \qquad \triangleleft$$

Integrals of tangents and secants

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Evaluate $I = \int \sec^3(x)\,dx.$

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Evaluate $I = \int \sec^3(x)\, dx$.

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Where we used that $\sec^2(x) = \tan'(x)$.

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Recall: $\sec'(x) = \dfrac{\sin(x)}{\cos^2(x)}$

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Recall: $\sec'(x) = \dfrac{\sin(x)}{\cos^2(x)} = \sec(x)\,\tan(x)$.

Integrals of tangents and secants

Example

Evaluate $I = \int \sec^3(x)\,dx$.

Solution: $I = \sec(x)\tan(x) - \int \tan(x)\,\sec'(x)\,dx$, and we also know $\sec'(x) = \sec(x)\tan(x)$.

Integrals of tangents and secants

Example

Evaluate $I = \int \sec^3(x)\, dx$.

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$$I = \sec(x)\tan(x) - \int \sec(x)\ \tan^2(x)\ dx$$

Integrals of tangents and secants

Example

Evaluate $I = \int \sec^3(x)\, dx$.

Solution: $I = \sec(x)\tan(x) - \int \tan(x)\, \sec'(x)\, dx$, and we also know $\sec'(x) = \sec(x)\tan(x)$.

$$I = \sec(x)\tan(x) - \int \sec(x)\, \tan^2(x)\, dx$$

$$I = \sec(x)\tan(x) - \int \sec(x)\left(\sec^2(x) - 1\right)\, dx$$

Integrals of tangents and secants

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$$\int \sec^3(x)\, dx = \sec(x)\tan(x) + \int \sec(x)\, dx - \int \sec^3(x)\, dx.$$

Integrals of tangents and secants

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$$\int \sec^3(x)\,dx = \sec(x)\tan(x) + \int \sec(x)\,dx - \int \sec^3(x)\,dx.$$

$$\int \sec^3(x)\,dx = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln\big(\sec(x) + \tan(x)\big) + c.$$

Integrals of tangents and secants

Recall: $\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$

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Proof:

$$I = \int \sec(x)\,dx$$

Integrals of tangents and secants

Recall: $\displaystyle\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$

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$$I = \int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$$

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Proof:

$$I = \int \sec(x)\,dx = \int \frac{1}{\cos(x)}\,dx$$

$$I = \int \frac{1}{\cos(x)}\,\frac{[1 + \sin(x)]}{\cos(x)}\,\frac{\cos(x)}{[1 + \sin(x)]}\,dx$$

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$$I = \int \frac{[1+\sin(x)]}{\cos^2(x)}\,\frac{1}{\Big(\frac{[1+\sin(x)]}{\cos(x)}\Big)}\,dx$$

$$I = \int \Big(\frac{[1+\sin(x)]}{\cos(x)}\Big)'\,\frac{1}{\Big(\frac{[1+\sin(x)]}{\cos(x)}\Big)}\,dx$$

Integrals of tangents and secants

Recall: $\int \sec(x)\, dx = \ln\big(\sec(x) + \tan(x)\big) + c.$

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Integrals of tangents and secants

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Substitute $u = \sec(x) + \tan(x)$, then

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So we obtain the formula,

$$\int \sec(x)\,dx = \ln\big(\sec(x) + \tan(x)\big) + c.$$

Trigonometric integrals (Sect. 8.2)

- Product of sines and cosines.
- Eliminating square roots.
- Integrals of tangents and secants.
- **Products of sines and cosines.**

Products of sines and cosines

Remark: The identities

$$\sin(\theta)\,\sin(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) - \cos(\theta+\phi)\big]$$
$$\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta-\phi) + \sin(\theta+\phi)\big]$$
$$\cos(\theta)\,\cos(\phi) = \frac{1}{2}\big[\cos(\theta-\phi) + \cos(\theta+\phi)\big].$$

can be used to compute integrals of the form

$$\int \sin(mx)\,\sin(nx)\,dx, \qquad \int \sin(mx)\,\cos(nx)\,dx,$$

$$\int \cos(mx)\,\cos(nx)\,dx.$$

Products of sines and cosines

Example

Evaluate: $I = \int \sin(3x)\, \cos(4x)\, dx.$

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Evaluate: $I = \int \sin(3x)\,\cos(4x)\,dx.$

Solution: Recall: $\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta - \phi) + \sin(\theta + \phi)\big].$

Products of sines and cosines

Example

Evaluate: $I = \int \sin(3x)\,\cos(4x)\,dx.$

Solution: Recall: $\sin(\theta)\,\cos(\phi) = \frac{1}{2}\big[\sin(\theta-\phi)+\sin(\theta+\phi)\big].$

The formula above implies,

$$I = \frac{1}{2}\int \big[\sin((3-4)x)+\sin((3+4)x)\big]\,dx,$$

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that is,

$$I = \frac{1}{2}\int \big[-\sin(x) + \sin(7x)\big]\,dx.$$

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The formula above implies,

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that is,

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This integral is simple to do,

$$I = \frac{1}{2}\Big[\cos(x) - \frac{1}{7}\,\cos(7x)\Big] + c. \qquad \triangleleft$$

Trigonometric substitutions (Sect. 8.3)

- Substitutions to cancel the square root
- Integrals involving $\sqrt{a^2 - x^2}$: Use $x = a\sin(\theta)$.
- Integrals involving $\sqrt{a^2 + x^2}$: Use $x = a\tan(\theta)$.
- Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.

Substitutions to cancel the square root

Remark: Integrals involving $\sqrt{a^2 - x^2}$ can be found with the substitution $x = a\sin(\theta)$.

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Notice: The substitution $x = a\cos(\theta)$ also works.

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Notice: The substitution $x = a\cos(\theta)$ also works.

Remark: We have used Pythagoras Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

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Notice: The substitution $x = a\cos(\theta)$ also works.

Remark: We have used Pythagoras Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

To compute dx we will need the following derivatives:

$$\sin'(\theta) = \cos(\theta), \qquad \cos'(\theta) = -\sin(\theta).$$

Substitutions to cancel the square root

Recall:

$$\sec^2(\theta) = \tan^2(\theta) + 1 = \tan'(\theta), \quad \sec'(\theta) = \sec(\theta)\,\tan(\theta).$$

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Remark: Integrals involving $\sqrt{x^2 - a^2}$ can be found with the substitution $x = a\sec(\theta)$.

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Trigonometric substitutions (Sect. 8.3)

- Substitutions to cancel the square root
- **Integrals involving $\sqrt{a^2 - x^2}$: Use $x = a\sin(\theta)$.**
- Integrals involving $\sqrt{a^2 + x^2}$: Use $x = a\tan(\theta)$.
- Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.

Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-5}^{5} \sqrt{25 - x^2}\, dx$.

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Evaluate $I = \int_{-5}^{5} \sqrt{25 - x^2}\, dx$.

Solution: Recall Pythagoras Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

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Evaluate $I = \int_{-5}^{5} \sqrt{25 - x^2}\, dx$.

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$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \Rightarrow \quad |\cos(\theta)| = \sqrt{1 - \sin^2(\theta)}.$$

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$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \Rightarrow \quad |\cos(\theta)| = \sqrt{1 - \sin^2(\theta)}.$$

Substitute: $x = 5\sin(\theta)$,

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$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \Rightarrow \quad |\cos(\theta)| = \sqrt{1 - \sin^2(\theta)}.$$

Substitute: $x = 5\sin(\theta)$, then $dx = 5\cos(\theta)\, d\theta$. Hence,

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - 25\sin^2(\theta)}\, \big(5\,\cos(\theta)\, d\theta\big),$$

Integrals involving $\sqrt{a^2 - x^2}$

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Substitute: $x = 5\sin(\theta)$, then $dx = 5\cos(\theta)\, d\theta$. Hence,

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - 25\sin^2(\theta)}\, \big(5\, \cos(\theta)\, d\theta\big),$$

$$I = 5 \int_{-\pi/2}^{\pi/2} \sqrt{25\big[1 - \sin^2(\theta)\big]}\, \cos(\theta)\, d\theta,$$

Integrals involving $\sqrt{a^2 - x^2}$

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Substitute: $x = 5\sin(\theta)$, then $dx = 5\cos(\theta)\, d\theta$. Hence,

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - 25\sin^2(\theta)}\, \big(5\,\cos(\theta)\, d\theta\big),$$

$$I = 5 \int_{-\pi/2}^{\pi/2} \sqrt{25\big[1 - \sin^2(\theta)\big]}\, \cos(\theta)\, d\theta,$$

$$I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta.$$

Integrals involving $\sqrt{a^2 - x^2}$

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Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx$.

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Evaluate $I = \displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx.$

Solution: Recall: $I = 5^2 \displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta.$

$$I = 5^2 \int_{-\pi/2}^{\pi/2} |\cos(\theta)|\, \cos(\theta)\, dx = 5^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx$.

Solution: Recall: $I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta.$

$$I = 5^2 \int_{-\pi/2}^{\pi/2} |\cos(\theta)|\, \cos(\theta)\, dx = 5^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

$$I = \frac{25}{2} \int_{-\pi/2}^{\pi/2} \big[1 + \cos(2\theta)\big]\, d\theta$$

Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx$.

Solution: Recall: $I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta$.

$$I = 5^2 \int_{-\pi/2}^{\pi/2} |\cos(\theta)|\, \cos(\theta)\, dx = 5^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

$$I = \frac{25}{2} \int_{-\pi/2}^{\pi/2} \big[1 + \cos(2\theta)\big]\, d\theta = \frac{25}{2} \Big(\theta + \frac{\sin(2\theta)}{2}\Big)\Big|_{-\pi/2}^{\pi/2}.$$

Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx$.

Solution: Recall: $I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta$.

$$I = 5^2 \int_{-\pi/2}^{\pi/2} |\cos(\theta)|\, \cos(\theta)\, dx = 5^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

$$I = \frac{25}{2} \int_{-\pi/2}^{\pi/2} \big[1 + \cos(2\theta)\big]\, d\theta = \frac{25}{2} \Big(\theta + \frac{\sin(2\theta)}{2}\Big)\Big|_{-\pi/2}^{\pi/2}.$$

We conclude that $I = \dfrac{25\pi}{2}$. $\triangleleft$

Trigonometric substitutions (Sect. 8.3)

- Substitutions to cancel the square root
- Integrals involving $\sqrt{a^2 - x^2}$: Use $x = a\sin(\theta)$.
- **Integrals involving $\sqrt{a^2 + x^2}$: Use $x = a\tan(\theta)$.**
- Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2 + 4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\, dx}{\sqrt{x^2 + 4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3 \, dx}{\sqrt{x^2 + 4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\,d\theta$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $\displaystyle I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\,d\theta$, that is $dx = 2\sec^2(\theta)\,d\theta$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3 \, dx}{\sqrt{x^2 + 4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\, d\theta$, that is $dx = 2\sec^2(\theta)\, d\theta$.

$$I = \int \frac{2^3 \tan^3(\theta)}{\sqrt{4\tan^2(\theta) + 4}} \, 2\sec^2(\theta)\, d\theta$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $\displaystyle I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\,d\theta$, that is $dx = 2\sec^2(\theta)\,d\theta$.

$$I = \int \frac{2^3\tan^3(\theta)}{\sqrt{4\tan^2(\theta)+4}}\,2\sec^2(\theta)\,d\theta = 16\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sqrt{4\,[\tan^2(\theta)+1]}}\,d\theta.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\, dx}{\sqrt{x^2 + 4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\, d\theta$, that is $dx = 2\sec^2(\theta)\, d\theta$.

$$I = \int \frac{2^3 \tan^3(\theta)}{\sqrt{4\tan^2(\theta) + 4}}\, 2\sec^2(\theta)\, d\theta = 16 \int \frac{\tan^3(\theta)\, \sec^2(\theta)}{\sqrt{4\,[\tan^2(\theta) + 1]}}\, d\theta.$$

$$I = \frac{16}{2} \int \frac{\tan^3(\theta)\, \sec^2(\theta)}{|\sec(\theta)|}\, d\theta$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\,d\theta$, that is $dx = 2\sec^2(\theta)\,d\theta$.

$$I = \int \frac{2^3 \tan^3(\theta)}{\sqrt{4\tan^2(\theta)+4}}\, 2\sec^2(\theta)\,d\theta = 16 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sqrt{4\,[\tan^2(\theta)+1]}}\,d\theta.$$

$$I = \frac{16}{2} \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{|\sec(\theta)|}\,d\theta = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \displaystyle\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3 \, dx}{\sqrt{x^2 + 4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta) \sec^2(\theta)}{\sec(\theta)} \, d\theta$.

$$I = 8 \int \tan^3(\theta) \sec(\theta) \, d\theta$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\, dx}{\sqrt{x^2 + 4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\, \sec^2(\theta)}{\sec(\theta)}\, d\theta$.

$$I = 8 \int \tan^3(\theta)\, \sec(\theta)\, d\theta = 8 \int \tan^2(\theta)\, \tan(\theta) \sec(\theta)\, d\theta.$$

Now recall $\tan(\theta) \sec(\theta) = \sec'(\theta)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Now recall $\tan(\theta)\sec(\theta) = \sec'(\theta)$, and that $\tan^2(\theta) = \sec^2(\theta) - 1$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Now recall $\tan(\theta)\sec(\theta) = \sec'(\theta)$, and that $\tan^2(\theta) = \sec^2(\theta) - 1$.

$$I = 8 \int \left[\sec^2(\theta) - 1\right]\,\sec'(\theta)\,d\theta.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Now recall $\tan(\theta)\sec(\theta) = \sec'(\theta)$, and that $\tan^2(\theta) = \sec^2(\theta) - 1$.

$$I = 8 \int \left[\sec^2(\theta) - 1\right]\,\sec'(\theta)\,d\theta.$$

We now do a substitution: $u = \sec(\theta)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Now recall $\tan(\theta)\sec(\theta) = \sec'(\theta)$, and that $\tan^2(\theta) = \sec^2(\theta) - 1$.

$$I = 8 \int \left[\sec^2(\theta) - 1\right]\,\sec'(\theta)\,d\theta.$$

We now do a substitution: $u = \sec(\theta)$, hence $du = \sec'(\theta)\,d\theta$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8 \int (u^2 - 1)\,du$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\, dx}{\sqrt{x^2 + 4}}$.

Solution: Recall: $I = 8 \int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\, d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\, d\theta$.

$$I = 8 \int (u^2 - 1)\, du = 8\left(\frac{u^3}{3} - u + c\right).$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8\int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8\int (u^2 - 1)\,du = 8\Big(\frac{u^3}{3} - u + c\Big).$$

Substitute back $u = \sec(\theta)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8\int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8\int (u^2 - 1)\,du = 8\Big(\frac{u^3}{3} - u + c\Big).$$

Substitute back $u = \sec(\theta)$, that is,

$$I = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + c.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $\quad I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $\quad I = 8 \int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\,d\theta$ together with the substitution: $\quad u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8 \int (u^2 - 1)\,du = 8\Big(\frac{u^3}{3} - u + c\Big).$$

Substitute back $u = \sec(\theta)$, that is,

$$I = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + c.$$

Still substitute back $x = 2\tan(\theta)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \int \left[\sec^2(\theta) - 1\right] \sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8 \int (u^2 - 1)\,du = 8\left(\frac{u^3}{3} - u + c\right).$$

Substitute back $u = \sec(\theta)$, that is,

$$I = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + c.$$

Still substitute back $x = 2\tan(\theta)$, that is $\theta = \arctan(x/2)$,

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$.

Solution: Recall: $I = 8 \displaystyle\int \big[\sec^2(\theta) - 1\big]\,\sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8\int (u^2 - 1)\,du = 8\Big(\frac{u^3}{3} - u + c\Big).$$

Substitute back $u = \sec(\theta)$, that is,

$$I = \frac{8}{3}\sec^3(\theta) - 8\sec(\theta) + c.$$

Still substitute back $x = 2\tan(\theta)$, that is $\theta = \arctan(x/2)$, hence

$$I = \frac{8}{3}\sec^3\big(\arctan(x/2)\big) - 8\sec(\arctan(x/2) + c. \qquad \triangleleft$$

Trigonometric substitutions (Sect. 8.3)

- Substitutions to cancel the square root
- Integrals involving $\sqrt{a^2 - x^2}$: Use $x = a\sin(\theta)$.
- Integrals involving $\sqrt{a^2 + x^2}$: Use $x = a\tan(\theta)$.
- **Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.**

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2 - 9}}$, for $x > 0$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \displaystyle\int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$,

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$,

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2 - 9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2 - 9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

$$I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\,\sqrt{\sec^2(\theta) - 1}}$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \displaystyle\int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

$$I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\,\sqrt{\sec^2(\theta) - 1}} = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)|\tan(\theta)|}.$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

$$I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\,\sqrt{\sec^2(\theta) - 1}} = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)|\tan(\theta)|}.$$

Recall: $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$,

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

The substitution is $x = 3\sec(\theta)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

$$I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\,\sqrt{\sec^2(\theta) - 1}} = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)|\tan(\theta)|}.$$

Recall: $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$, and $x > 0$ implies $\tan(\theta) > 0$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \displaystyle\int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \dfrac{2}{9}\displaystyle\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)}$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)}$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{2}{9}\int \cos(\theta)\,d\theta.$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{2}{9}\int \cos(\theta)\,d\theta.$$

$$I = \frac{2}{9}\sin(\theta) + c.$$

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{2}{9}\int \cos(\theta)\,d\theta.$$

$$I = \frac{2}{9}\sin(\theta) + c.$$

Substitute back $x = 3\sec(\theta)$,

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 0$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{2}{9}\int \cos(\theta)\,d\theta.$$

$$I = \frac{2}{9}\sin(\theta) + c.$$

Substitute back $x = 3\sec(\theta)$, that is, $\theta = \text{arcsec}(x/3)$.

Integrals involving $\sqrt{x^2 - a^2}$

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