Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

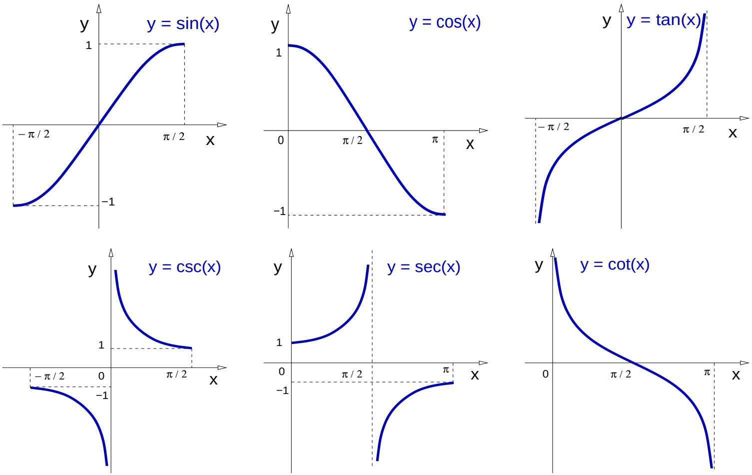
- Review: Definitions and properties.
- Derivatives.
- Integrals.

Last class: Definitions and properties.

- Domains restrictions and inverse trigs.
- Evaluating inverse trigs at simple values.
- Few identities for inverse trigs.

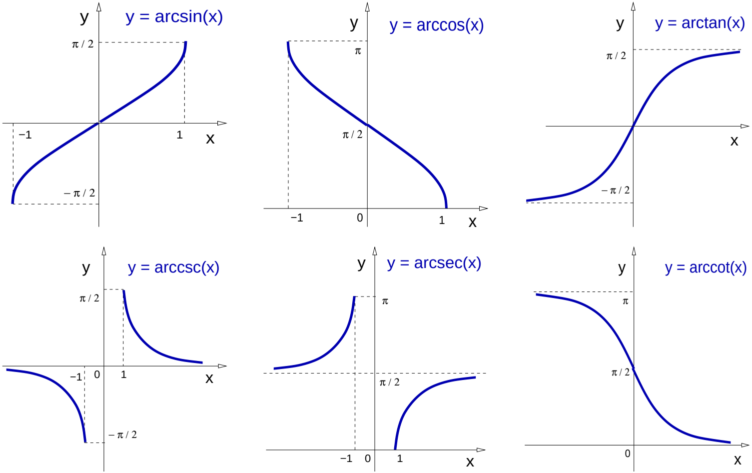
Review: Definitions and properties

Remark: On certain domains the trigonometric functions are invertible.



Review: Definitions and properties

Remark: The graph of the inverse function is a reflection of the original function graph about the $y = x$ axis.



Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arccos(x) + \arccos(-x) = \pi, \qquad \arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

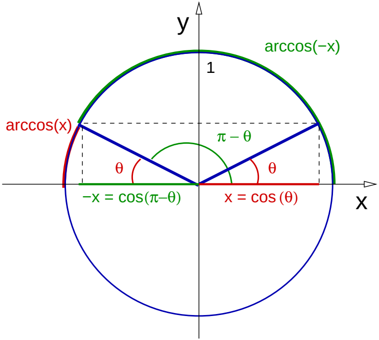
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arccos(x) + \arccos(-x) = \pi, \qquad \arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

Proof:



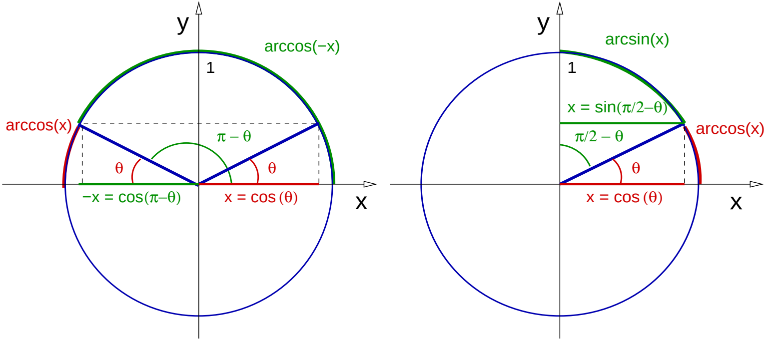
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arccos(x) + \arccos(-x) = \pi, \qquad \arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

Proof:



Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arcsin(-x) = -\arcsin(x),$$
$$\arctan(-x) = -\arctan(x),$$
$$\text{arccsc}(-x) = -\text{arccsc}(x).$$

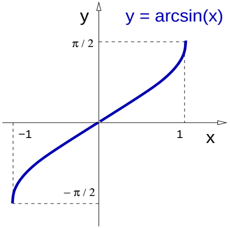
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arcsin(-x) = -\arcsin(x),$$
$$\arctan(-x) = -\arctan(x),$$
$$\text{arccsc}(-x) = -\text{arccsc}(x).$$

Proof:



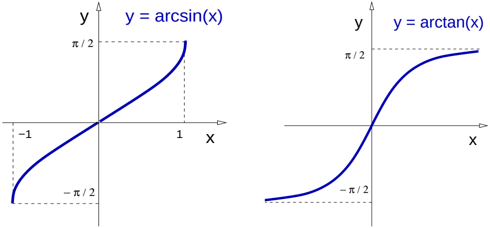
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arcsin(-x) = -\arcsin(x),$$
$$\arctan(-x) = -\arctan(x),$$
$$\text{arccsc}(-x) = -\text{arccsc}(x).$$

Proof:



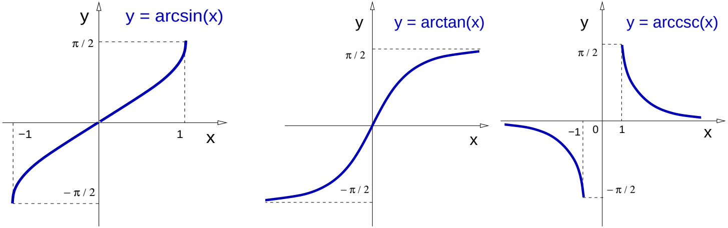
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ the following identities hold,

$$\arcsin(-x) = -\arcsin(x),$$
$$\arctan(-x) = -\arctan(x),$$
$$\text{arccsc}(-x) = -\text{arccsc}(x).$$

Proof:



Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

- Review: Definitions and properties.
- **Derivatives.**
- Integrals.

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$\big(f^{-1}\big)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$.

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)}$$

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)} = \frac{1}{\cos\big(\arcsin(x)\big)}$$

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\left(\arcsin(x)\right)} = \frac{1}{\cos\left(\arcsin(x)\right)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$,

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\left(\arcsin(x)\right)} = \frac{1}{\cos\left(\arcsin(x)\right)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the cosine is positive in that interval,

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$\big(f^{-1}\big)'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)} = \frac{1}{\cos\big(\arcsin(x)\big)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \Big[\frac{\pi}{2}, \frac{\pi}{2}\Big]$, and the cosine is positive in that interval, then $\cos(y) = +\sqrt{1 - \sin^2(y)}$,

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)} = \frac{1}{\cos\big(\arcsin(x)\big)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the cosine is positive in that interval, then $\cos(y) = +\sqrt{1-\sin^2(y)}$, hence

$$\arcsin'(x) = \frac{1}{\sqrt{1-\sin^2\big(\arcsin(x)\big)}}$$

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)} = \frac{1}{\cos\big(\arcsin(x)\big)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \Big[\dfrac{\pi}{2}, \dfrac{\pi}{2}\Big]$, and the cosine is positive in that interval, then $\cos(y) = +\sqrt{1 - \sin^2(y)}$, hence

$$\arcsin'(x) = \frac{1}{\sqrt{1 - \sin^2\big(\arcsin(x)\big)}} \;\Rightarrow\; \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}. \qquad \square$$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\begin{aligned}
\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}}, & \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}, & |x| &\leqslant 1,\\
\arctan'(x) &= \frac{1}{1+x^2}, & \operatorname{arccot}'(x) &= -\frac{1}{1+x^2}, & x &\in \mathbb{R},\\
\operatorname{arcsec}'(x) &= \frac{1}{|x|\sqrt{x^2-1}}, & \operatorname{arccsc}'(x) &= -\frac{1}{|x|\sqrt{x^2-1}}, & |x| &\geqslant 1.
\end{aligned}$$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)},$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}, \;\; \tan'(y) = \dfrac{\cos^2(y)+\sin^2(y)}{\cos^2(y)}$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}, \quad \tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

$$\tan'(y) = 1 + \tan^2(y),$$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}, \;\; \tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

$$\tan'(y) = 1 + \tan^2(y), \;\; y = \arctan(x),$$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \quad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}, \;\; \tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

$$\tan'(y) = 1 + \tan^2(y), \;\; y = \arctan(x), \quad \Rightarrow \quad \arctan'(x) = \frac{1}{1+x^2}.$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\left(\operatorname{arcsec}(x)\right)}$,

Derivatives of inverse trigonometric functions

Proof: $\text{arcsec}'(x) = \dfrac{1}{\sec'\big(\text{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$.

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)'$$

Derivatives of inverse trigonometric functions

Proof: $\text{arcsec}'(x) = \dfrac{1}{\sec'\big(\text{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \text{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)},$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \qquad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \left(\frac{1}{\cos(y)}\right)' = \frac{\sin(y)}{\cos^2(y)}, \quad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)}$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \qquad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)} = \frac{1}{|\cos(y)|}\,\frac{\sqrt{1 - \cos^2(y)}}{|\cos(y)|},$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \qquad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)} = \frac{1}{|\cos(y)|}\,\frac{\sqrt{1 - \cos^2(y)}}{|\cos(y)|},$$

$$\sec'(y) = \frac{1}{|\cos(y)|}\sqrt{\frac{1}{\cos^2(y)} - 1}$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \qquad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)} = \frac{1}{|\cos(y)|}\,\frac{\sqrt{1 - \cos^2(y)}}{|\cos(y)|},$$

$$\sec'(y) = \frac{1}{|\cos(y)|}\sqrt{\frac{1}{\cos^2(y)} - 1} = |\sec(y)|\,\sqrt{\sec^2(y) - 1}.$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \qquad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)} = \frac{1}{|\cos(y)|}\,\frac{\sqrt{1 - \cos^2(y)}}{|\cos(y)|},$$

$$\sec'(y) = \frac{1}{|\cos(y)|}\sqrt{\frac{1}{\cos^2(y)} - 1} = |\sec(y)|\,\sqrt{\sec^2(y) - 1}.$$

We conclude: $\operatorname{arcsec}'(x) = \dfrac{1}{|x|\,\sqrt{x^2 - 1}}$. $\square$

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x+7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2-1}}$.

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies,

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7|\,\sqrt{(3x + 7)^2 - 1}}$. $\triangleleft$

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7|\,\sqrt{(3x + 7)^2 - 1}}$. $\lhd$

Example

Compute the derivative of $y(x) = \arctan(4\ln(x))$.

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \text{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\text{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7| \sqrt{(3x + 7)^2 - 1}}$. $\lhd$

Example

Compute the derivative of $y(x) = \arctan(4\ln(x))$.

Solution: Recall the main formula: $\arctan'(u) = \dfrac{1}{1 + u^2}$.

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7|\,\sqrt{(3x + 7)^2 - 1}}$. $\lhd$

Example

Compute the derivative of $y(x) = \arctan(4 \ln(x))$.

Solution: Recall the main formula: $\arctan'(u) = \dfrac{1}{1 + u^2}$.

Therefore, chain rule implies,

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7|\,\sqrt{(3x + 7)^2 - 1}}$. $\triangleleft$

Example

Compute the derivative of $y(x) = \arctan(4\ln(x))$.

Solution: Recall the main formula: $\arctan'(u) = \dfrac{1}{1 + u^2}$.

Therefore, chain rule implies,

$$y'(x) = \frac{1}{\left[1 + \big(4\ln(x)\big)^2\right]} \frac{4}{x}$$

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7| \sqrt{(3x + 7)^2 - 1}}$. $\triangleleft$

Example

Compute the derivative of $y(x) = \arctan(4\ln(x))$.

Solution: Recall the main formula: $\arctan'(u) = \dfrac{1}{1 + u^2}$.

Therefore, chain rule implies,

$$y'(x) = \frac{1}{\left[1 + \left(4\ln(x)\right)^2\right]} \frac{4}{x} \quad \Rightarrow \quad y' = \frac{4}{x\left[1 + 16\ln^2(x)\right]}. \quad \triangleleft$$

Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

- Review: Definitions and properties.
- Derivatives.
- **Integrals.**

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\,\arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\mathrm{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.)

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a} \operatorname{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\left(\frac{x}{a}\right) + c,$

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\left(\frac{x}{a}\right) + c$, then

$y'(x)$

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\text{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\left(\frac{x}{a}\right) + c$, then

$$y'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\,\frac{1}{a}$$

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\Big(\frac{x}{a}\Big) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\Big(\frac{x}{a}\Big) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\operatorname{arcsec}\Big(\Big|\frac{x}{a}\Big|\Big) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\Big(\frac{x}{a}\Big) + c$, then

$$y'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\,\frac{1}{a} = \frac{|a|}{\sqrt{a^2 - x^2}}\,\frac{1}{a}$$

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\left(\frac{x}{a}\right) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\,\arctan\left(\frac{x}{a}\right) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\text{arcsec}\left(\left|\frac{x}{a}\right|\right) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\left(\frac{x}{a}\right) + c$, then

$$y'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\,\frac{1}{a} = \frac{|a|}{\sqrt{a^2 - x^2}}\,\frac{1}{a} \;\Rightarrow\; y'(x) = \frac{1}{\sqrt{a^2 - x^2}}$$

□

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}} \, dx.$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$,

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3 - u^2}} \frac{du}{2}$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3-u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3-u^2}}.$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3-u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3-u^2}}.$$

Recall: $\int \frac{dx}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + c.$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3 - u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3 - u^2}}.$$

Recall: $\int \frac{dx}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + c.$ Then, for $a = \sqrt{3}$,

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3 - u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3 - u^2}}.$$

Recall: $\int \frac{dx}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + c.$ Then, for $a = \sqrt{3}$,

$$I = 3 \arcsin\left(\frac{u}{\sqrt{3}}\right) + c$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3 - u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3 - u^2}}.$$

Recall: $\displaystyle\int \frac{dx}{\sqrt{a^2 - u^2}} = \arcsin\left(\frac{u}{a}\right) + c.$ Then, for $a = \sqrt{3}$,

$$I = 3 \arcsin\left(\frac{u}{\sqrt{3}}\right) + c \quad \Rightarrow \quad I = 3 \arcsin\left(\frac{2(x-1)}{\sqrt{3}}\right) + c. \quad \triangleleft$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$,

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\,dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

$$I = \int \frac{6}{t\left[\ln^2(t) + 4\ln(t) + 8\right]}\, dt,$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\,dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

$$I = \int \frac{6}{t\left[\ln^2(t) + 4\ln(t) + 8\right]}\,dt,$$

$$I = \int \frac{6}{t\left[\ln^2(t) + 2(2\ln(t)) + 4 - 4 + 8\right]}\,dt$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

$$I = \int \frac{6}{t\left[\ln^2(t) + 4\ln(t) + 8\right]}\, dt,$$

$$I = \int \frac{6}{t\left[\ln^2(t) + 2(2\ln(t)) + 4 - 4 + 8\right]}\, dt$$

$$I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

$$I = \int \frac{6}{t\left[\ln^2(t) + 4\ln(t) + 8\right]}\, dt,$$

$$I = \int \frac{6}{t\left[\ln^2(t) + 2(2\ln(t)) + 4 - 4 + 8\right]}\, dt$$

$$I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt$$

This looks like the derivative of the arctangent.

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \displaystyle\int \frac{6}{t\left[\big(\ln(t) + 2\big)^2 + 4\right]}\, dt.$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\,dt.$

Solution: Recall: $I = \displaystyle\int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\,dt.$

Substitute: $u = \ln(t) + 2,$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \displaystyle\int \frac{6}{t\left[\big(\ln(t) + 2\big)^2 + 4\right]}\, dt.$

Substitute: $u = \ln(t) + 2$, then $du = \dfrac{1}{t}\, dt,$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\,dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\,dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\,dt,$

$$I = \int \frac{6}{4 + u^2}\,du$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\, dt$,

$$I = \int \frac{6}{4 + u^2}\, du = 6 \int \frac{du}{2^2 + u^2}$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\, dt,$

$$I = \int \frac{6}{4 + u^2}\, du = 6 \int \frac{du}{2^2 + u^2} = 6\, \frac{1}{2}\, \arctan\left(\frac{u}{2}\right) + c.$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\,dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\,dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\,dt$,

$$I = \int \frac{6}{4 + u^2}\,du = 6 \int \frac{du}{2^2 + u^2} = 6\,\frac{1}{2}\,\arctan\left(\frac{u}{2}\right) + c.$$

$I = 3\,\arctan\left(\frac{1}{2}(\ln(t) + 2)\right) + c$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\, dt$,

$$I = \int \frac{6}{4 + u^2}\, du = 6 \int \frac{du}{2^2 + u^2} = 6\, \frac{1}{2}\, \arctan\left(\frac{u}{2}\right) + c.$$

$I = 3\, \arctan\left(\frac{1}{2}(\ln(t) + 2)\right) + c \quad \Rightarrow \quad I = 3\, \arctan\left(\ln(\sqrt{t}) + 1\right) + c.$

$\triangleleft$

Hyperbolic functions (Sect. 7.7)

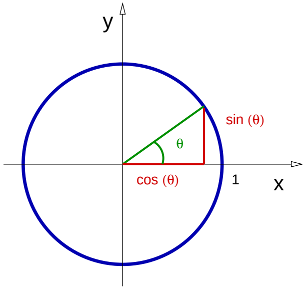
- Circular and hyperbolic functions.
- Definitions and identities.
- Derivatives of hyperbolic functions.
- Integrals of hyperbolic functions.

Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.

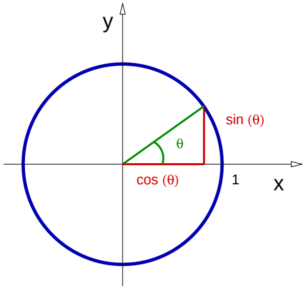
Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.



Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.

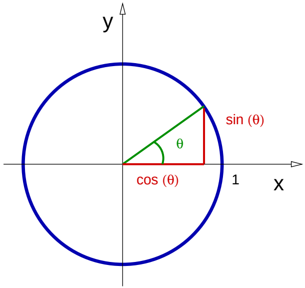


The circle $x^2 + y^2 = 1$ can be parametrized by the functions

$$x = \cos(\theta),$$
$$y = \sin(\theta).$$

Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.



The circle $x^2 + y^2 = 1$ can be parametrized by the functions

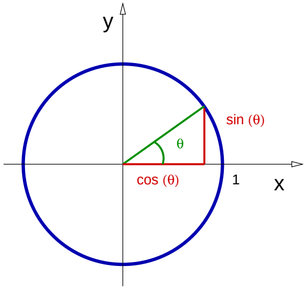
$$x = \cos(\theta),$$
$$y = \sin(\theta).$$

Since these functions satisfy

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.



The circle $x^2 + y^2 = 1$ can be parametrized by the functions

$$x = \cos(\theta),$$
$$y = \sin(\theta).$$

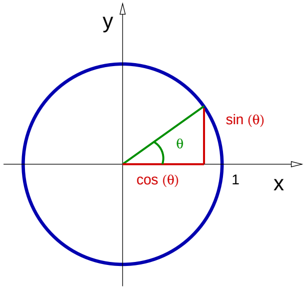
Since these functions satisfy

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Remark: The parametrization is not unique.

Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.



The circle $x^2 + y^2 = 1$ can be parametrized by the functions

$$x = \cos(\theta),$$
$$y = \sin(\theta).$$

Since these functions satisfy

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

Remark: The parametrization is not unique. Another solution is

$$x = \cos(n\theta), \qquad y = \sin(n\theta), \qquad n \in \mathbb{N}.$$

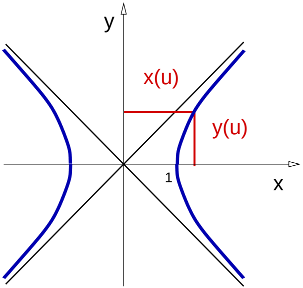
Circular and hyperbolic functions

Remark:

Hyperbolic functions are a parametrization of a hyperbola.

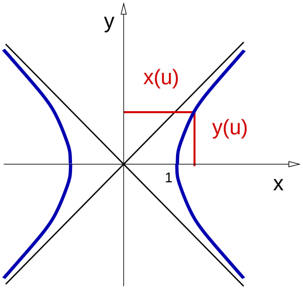
Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



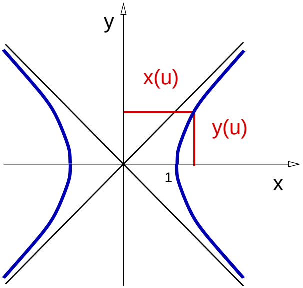
The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

Circular and hyperbolic functions

Remark:

Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

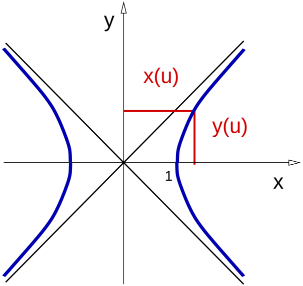
$$x = f(u), \quad y = g(u),$$

satisfying the condition

$$f^2(u) - g^2(u) = 1.$$

Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

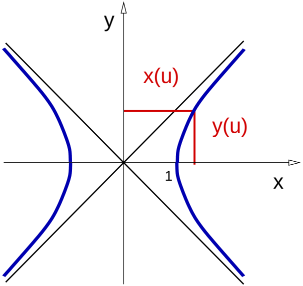
satisfying the condition

$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big]$, $y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big]$,

Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

satisfying the condition

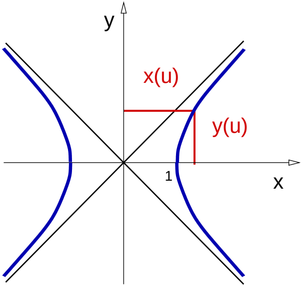
$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big]$, $y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big]$,

$$x^2 - y^2 =$$

Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

satisfying the condition

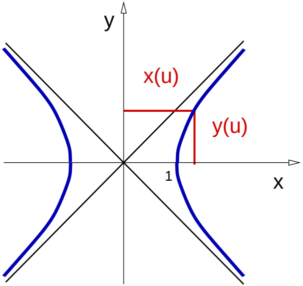
$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big]$, $y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big]$,

$$x^2 - y^2 = \frac{1}{4}\Big[h^2 + \frac{1}{h^2} + 2$$

Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

satisfying the condition

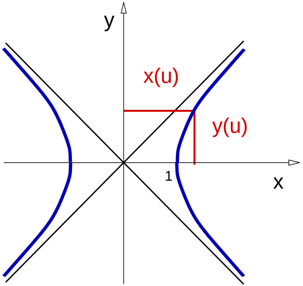
$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big]$, $y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big]$,

$$x^2 - y^2 = \frac{1}{4}\Big[h^2 + \frac{1}{h^2} + 2 - h^2 - \frac{1}{h^2} + 2\Big]$$

Circular and hyperbolic functions

Remark:
Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

satisfying the condition

$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \; y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$

$$x^2 - y^2 = \frac{1}{4}\Big[h^2 + \frac{1}{h^2} + 2 - h^2 - \frac{1}{h^2} + 2\Big] = 1.$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \qquad y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\left[h(u) + \frac{1}{h(u)}\right], \qquad y = \frac{1}{2}\left[h(u) - \frac{1}{h(u)}\right],$$

where h is any non-zero continuous function satisfying

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\left[h(u) + \frac{1}{h(u)}\right], \qquad y = \frac{1}{2}\left[h(u) - \frac{1}{h(u)}\right],$$

where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty,$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \qquad y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$$

where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to-\infty} h(u) = 0,$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \qquad y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$$

  where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to-\infty} h(u) = 0, \qquad h(0) = 1.$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\left[h(u) + \frac{1}{h(u)}\right], \qquad y = \frac{1}{2}\left[h(u) - \frac{1}{h(u)}\right],$$

  where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to-\infty} h(u) = 0, \qquad h(0) = 1.$$

- The hyperbolic trigonometric functions correspond to

$$h(u) = e^u.$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \qquad y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$$

  where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to-\infty} h(u) = 0, \qquad h(0) = 1.$$

- The hyperbolic trigonometric functions correspond to

$$h(u) = e^u.$$

Definition

The *hyperbolic trigonometric functions* are defined by

$$\cosh(u) = \frac{e^u + e^{-u}}{2}, \qquad \sinh(u) = \frac{e^u - e^{-u}}{2}.$$

Hyperbolic functions (Sect. 7.7)

- Circular and hyperbolic functions.
- **Definitions and identities.**
- Derivatives of hyperbolic functions.
- Integrals of hyperbolic functions.

Definitions and identities

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Definitions and identities

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Remarks:

- These functions satisfy identities similar but not equal to those satisfied by circular trigonometric functions.

Definitions and identities

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Remarks:

- These functions satisfy identities similar but not equal to those satisfied by circular trigonometric functions.
- We have seen one of these identities:

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big]$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big] = \frac{1}{2}\Big[\Big(e^{x}\Big)^2 - \Big(\frac{1}{e^{x}}\Big)^2\Big].$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big] = \frac{1}{2}\Big[\Big(e^x\Big)^2 - \Big(\frac{1}{e^x}\Big)^2\Big].$$

Recalling the formula $a^2 - b^2 = (a+b)(a-b)$,

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big] = \frac{1}{2}\Big[\Big(e^x\Big)^2 - \Big(\frac{1}{e^x}\Big)^2\Big].$$

Recalling the formula $a^2 - b^2 = (a+b)(a-b)$,

$$\sinh(2x) = \frac{2}{4}\Big[e^x + \frac{1}{e^x}\Big]\Big[e^x - \frac{1}{e^x}\Big]$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big] = \frac{1}{2}\Big[\big(e^x\big)^2 - \Big(\frac{1}{e^x}\Big)^2\Big].$$

Recalling the formula $a^2 - b^2 = (a+b)(a-b)$,

$$\sinh(2x) = \frac{2}{4}\Big[e^x + \frac{1}{e^x}\Big]\Big[e^x - \frac{1}{e^x}\Big] = 2\cosh(x)\sinh(x).$$

□

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7))$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right]$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right]$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \frac{25}{7}$.

$$\sinh(2\ln(3))$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \frac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\left[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\right]$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\left[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\right] = \frac{1}{2}\left[e^{\ln(9)} - \frac{1}{e^{\ln(9)}}\right]$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\left[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\right] = \frac{1}{2}\left[e^{\ln(9)} - \frac{1}{e^{\ln(9)}}\right]$$

$$\sinh(2\ln(3)) = \frac{1}{2}\left[9 - \frac{1}{9}\right]$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\left[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\right] = \frac{1}{2}\left[e^{\ln(9)} - \frac{1}{e^{\ln(9)}}\right]$$

$$\sinh(2\ln(3)) = \frac{1}{2}\left[9 - \frac{1}{9}\right] = \frac{1}{2}\,\frac{80}{9}$$

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\left[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\right] = \frac{1}{2}\left[7 + \frac{1}{7}\right] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\left[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\right] = \frac{1}{2}\left[e^{\ln(9)} - \frac{1}{e^{\ln(9)}}\right]$$

$$\sinh(2\ln(3)) = \frac{1}{2}\left[9 - \frac{1}{9}\right] = \frac{1}{2}\,\frac{80}{9} \quad\Rightarrow\quad \sinh(2\ln(3)) = \frac{40}{9}. \triangleleft$$

Hyperbolic functions (Sect. 7.7)

- Circular and hyperbolic functions.
- Definitions and identities.
- **Derivatives of hyperbolic functions.**
- Integrals of hyperbolic functions.

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

$$\sinh'(x) = \frac{1}{2}\big(e^x - e^{-x}\big)'$$

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

$$\sinh'(x) = \frac{1}{2}\big(e^x - e^{-x}\big)' = \frac{1}{2}\big(e^x - e^{-x}(-1)\big)$$

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

$$\sinh'(x) = \frac{1}{2}\big(e^x - e^{-x}\big)' = \frac{1}{2}\big(e^x - e^{-x}(-1)\big)$$

$$\sinh'(u) = \frac{1}{2}\big(e^x + e^{-x}\big)$$

Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

$$\sinh'(x) = \frac{1}{2}\left(e^x - e^{-x}\right)' = \frac{1}{2}\left(e^x - e^{-x}(-1)\right)$$

$$\sinh'(u) = \frac{1}{2}\left(e^x + e^{-x}\right) \quad \Rightarrow \quad \sinh'(x) = \cosh(x).$$

□

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)} \tanh'(3x)\, 3.$$

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)} \tanh'(3x)\, 3.$$

We only need to remember the first two formulas in the Theorem above,

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)}\,\tanh'(3x)\,3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \Big(\frac{\sinh(x)}{\cosh(x)}\Big)'$$

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)}\, \tanh'(3x)\, 3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \Big(\frac{\sinh(x)}{\cosh(x)}\Big)' = \frac{\sinh'(x)\, \cosh(x) - \sinh(x)\, \cosh'(x)}{\cosh^2(x)}$$

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)} \tanh'(3x)\, 3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \Big(\frac{\sinh(x)}{\cosh(x)}\Big)' = \frac{\sinh'(x) \cosh(x) - \sinh(x) \cosh'(x)}{\cosh^2(x)}$$

$$\tanh'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)}$$

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)}\,\tanh'(3x)\,3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \Big(\frac{\sinh(x)}{\cosh(x)}\Big)' = \frac{\sinh'(x)\,\cosh(x) - \sinh(x)\,\cosh'(x)}{\cosh^2(x)}$$

$$\tanh'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)}\,\tanh'(3x)\,3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \Big(\frac{\sinh(x)}{\cosh(x)}\Big)' = \frac{\sinh'(x)\,\cosh(x) - \sinh(x)\,\cosh'(x)}{\cosh^2(x)}$$

$$\tanh'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

We conclude that $y'(x) = \dfrac{3e^{\tanh(3x)}}{\cosh^2(3x)}$. $\triangleleft$

Hyperbolic functions (Sect. 7.7)

- Circular and hyperbolic functions.
- Definitions and identities.
- Derivatives of hyperbolic functions.
- **Integrals of hyperbolic functions.**

Integrals of hyperbolic functions

Theorem

For every real constant c the following expressions hold,

$$\int \sinh(x)\,dx = \cosh(x) + c, \quad \int \cosh(x)\,dx = \sinh(x) + c,$$

$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + c, \quad \int \operatorname{csch}^2(x)\,dx = -\coth(x) + c,$$

Integrals of hyperbolic functions

Theorem

For every real constant c the following expressions hold,

$$\int \sinh(x)\,dx = \cosh(x) + c, \quad \int \cosh(x)\,dx = \sinh(x) + c,$$
$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + c, \quad \int \operatorname{csch}^2(x)\,dx = -\coth(x) + c,$$

Proof: The derivative of each right-hand side above is the integrand in each left-hand side □

Integrals of hyperbolic functions

Theorem

For every real constant c the following expressions hold,

$$\int \sinh(x)\,dx = \cosh(x) + c, \quad \int \cosh(x)\,dx = \sinh(x) + c,$$
$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + c, \quad \int \operatorname{csch}^2(x)\,dx = -\coth(x) + c,$$

Proof: The derivative of each right-hand side above is the integrand in each left-hand side □

Remark: There are many other integration formulas, but the ones above are the most frequently used.

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\, dx$.

Solution: We try the substitution $u = 3x - \ln(2)$,

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\, dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\, dx$.

$$I = \int 6\cosh(u)\, \frac{du}{3}$$

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\,du$$

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\, dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\, dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\, du = 2\sinh(u) + c.$$

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\,du = 2\sinh(u) + c.$$

We conclude that $I = 2\sinh(3x - \ln(2)) + c$. $\triangleleft$

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\,du = 2\sinh(u) + c.$$

We conclude that $I = 2\sinh(3x - \ln(2)) + c$. $\triangleleft$

Remark: If needed, one can rewrite the sinh above as

$$\sinh(3x - \ln(2)) = \frac{1}{2}\left(e^{3x-\ln(2)} - e^{-3x+\ln(2)}\right)$$

Integrals of hyperbolic functions

Example

Evaluate $I = \int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\,du = 2\sinh(u) + c.$$

We conclude that $I = 2\sinh(3x - \ln(2)) + c$. ◁

Remark: If needed, one can rewrite the sinh above as

$$\sinh(3x - \ln(2)) = \frac{1}{2}\left(e^{3x-\ln(2)} - e^{-3x+\ln(2)}\right)$$

$$\sinh(3x - \ln(2)) = \frac{1}{2}\left(\frac{e^{3x}}{e^{\ln(2)}} - e^{-3x}\,e^{\ln(2)}\right) = \frac{e^{3x}}{4} - e^{-3x}.$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$.

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2),$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2)\, 6x \, dx.$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2)\, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6}$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2) \, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6} = \frac{4}{3} \int u^{-3} \, du$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2)\, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6} = \frac{4}{3} \int u^{-3} \, du = \frac{4}{3} \frac{u^{-2}}{(-2)} + c$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2)\, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6} = \frac{4}{3} \int u^{-3} \, du = \frac{4}{3} \, \frac{u^{-2}}{(-2)} + c = -\frac{2}{3} \, \frac{1}{u^2} + c$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx.$

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2) \, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6} = \frac{4}{3} \int u^{-3} \, du = \frac{4}{3} \, \frac{u^{-2}}{(-2)} + c = -\frac{2}{3} \, \frac{1}{u^2} + c$$

If we substitute back $u = \cosh(3x^2)$, we obtain

$$I = -\frac{2}{3} \, \frac{1}{\cosh^2(3x^2)} + c. \qquad \triangleleft$$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- Polynomial division.
- Multiplying by 1.

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives.

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ holds,

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives.
Let F be a primitive of f,

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$.

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u'$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ holds,

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ holds,

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Denoting $y = u(x)$, we get

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ holds,

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Denoting $y = u(x)$, we get

$$\int f(u(x))\, u'(x)\, dx = F(y)$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Denoting $y = u(x)$, we get

$$\int f(u(x))\, u'(x)\, dx = F(y) = \int F'(y)\, dy$$

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ *holds,*

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Denoting $y = u(x)$, we get

$$\int f(u(x))\, u'(x)\, dx = F(y) = \int F'(y)\, dy = \int f(y)\, dy. \qquad \square$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2,

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$.

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\,dx$. We try the substitution

$$u = \tan(2x^2),$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\,dx.$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$.

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1 + \cos(4x^2)]$.

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1 + \cos(4x^2)]$. And this is the case,

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\,dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\,dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1+\cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1+\cos(2\theta)\big),$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$.

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1 + \cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1 + \cos(2\theta)\big), \qquad \theta = 2x^2,$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$.

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1 + \cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1 + \cos(2\theta)\big), \qquad \theta = 2x^2,$$

$$I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$$

Substitution rule

Example

Evaluate $I = \displaystyle\int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1+\cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1+\cos(2\theta)\big), \qquad \theta = 2x^2,$$

$$I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx = \int \frac{3x\, e^{\tan(2x^2)}}{2\cos^2(2x^2)}\, dx$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$.

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\, (4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1 + \cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1 + \cos(2\theta)\big), \qquad \theta = 2x^2,$$

$$I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx = \int \frac{3x\, e^{\tan(2x^2)}}{2\cos^2(2x^2)}\, dx = \frac{3}{2} \int e^u\, \frac{du}{4}.$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: Recall: $I = \frac{3}{2} \int e^u \, \frac{du}{4}$, with $u = \tan(2x^2)$.

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx$.

Solution: Recall: $I = \frac{3}{2} \int e^u \frac{du}{4}$, with $u = \tan(2x^2)$.

$$I = \frac{3}{8} \int e^u\, du$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: Recall: $I = \frac{3}{2} \int e^u \, \frac{du}{4}$, with $u = \tan(2x^2)$.

$$I = \frac{3}{8} \int e^u \, du = \frac{3}{8}\, e^u + c.$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: Recall: $I = \frac{3}{2} \int e^u \, \frac{du}{4}$, with $u = \tan(2x^2)$.

$$I = \frac{3}{8} \int e^u \, du = \frac{3}{8}\, e^u + c.$$

We now substitute back with $u = \tan(2x^2)$,

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1 + \cos(4x^2)]}\, dx.$

Solution: Recall: $I = \frac{3}{2} \int e^u \, \frac{du}{4}$, with $u = \tan(2x^2)$.

$$I = \frac{3}{8} \int e^u \, du = \frac{3}{8}\, e^u + c.$$

We now substitute back with $u = \tan(2x^2)$,

$$I = \frac{3}{8}\, e^{\tan(2x^2)} + c.$$

◁

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- **Completing the square.**
- Trigonometric identities.
- Polynomial division.
- Multiplying by 1.

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}}$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \qquad u = x + 3,$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \quad u = x + 3, \quad du = dx$$

Completing the square

Example

Evaluate $I = \displaystyle\int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right) x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \qquad u = x + 3, \quad du = dx$$

$$I = \int \frac{du}{u\sqrt{u^2 - 5}}$$

Completing the square

Example

Evaluate $I = \displaystyle\int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \qquad u = x+3, \quad du = dx$$

$$I = \int \frac{du}{u\sqrt{u^2 - 5}} = \frac{1}{\sqrt{5}}\,\mathrm{arcsec}\left(\frac{|u|}{\sqrt{5}}\right) + c.$$

Completing the square

Example

Evaluate $I = \int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\left(\frac{6}{2}\right)x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \quad u = x+3, \quad du = dx$$

$$I = \int \frac{du}{u\sqrt{u^2 - 5}} = \frac{1}{\sqrt{5}} \operatorname{arcsec}\left(\frac{|u|}{\sqrt{5}}\right) + c.$$

We obtain $I = \frac{1}{\sqrt{5}} \operatorname{arcsec}\left(\frac{|x+3|}{\sqrt{5}}\right) + c.$ ◁

Completing the square

Remark: Sometimes completing the square is not needed.

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \displaystyle\int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator,

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4,$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \displaystyle\int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

$$I = \int \frac{1}{\sqrt{u}}\,\frac{du}{2}$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

$$I = \int \frac{1}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

$$I = \int \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2} \int u^{-1/2}\,du = \frac{1}{2}\left(2u^{1/2}\right) + c.$$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

$$I = \int \frac{1}{\sqrt{u}} \frac{du}{2} = \frac{1}{2} \int u^{-1/2}\,du = \frac{1}{2}\big(2u^{1/2}\big) + c.$$

We conclude that $I = \sqrt{x^2+6x+4} + c$. $\triangleleft$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- **Trigonometric identities.**
- Polynomial division.
- Multiplying by 1.

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 \, dx$.

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 \, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2 = \left[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right]^2$$

Trigonometric identities

Example

Evaluate $I = \int \big[\sec(x) + \tan(x)\big]^2 \, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \big[\sec(x) + \tan(x)\big]^2 = \Big[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\Big]^2 = \frac{(1+\sin(x))^2}{\cos^2(x)}.$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2 = \left[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right]^2 = \frac{(1+\sin(x))^2}{\cos^2(x)}.$$

$$f(x) = \frac{1 + 2\sin(x) + \sin^2(x)}{\cos^2(x)}$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2 = \left[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right]^2 = \frac{(1+\sin(x))^2}{\cos^2(x)}.$$

$$f(x) = \frac{1 + 2\sin(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}.$$

Trigonometric identities

Example

Evaluate $I = \displaystyle\int \left[\sec(x) + \tan(x)\right]^2 dx.$

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2 = \left[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right]^2 = \frac{(1 + \sin(x))^2}{\cos^2(x)}.$$

$$f(x) = \frac{1 + 2\sin(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}.$$

$$\frac{\sin^2(x)}{\cos^2(x)} = \frac{1 - \cos^2(x)}{\cos^2(x)}$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \left[\sec(x) + \tan(x)\right]^2 = \left[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right]^2 = \frac{(1+\sin(x))^2}{\cos^2(x)}.$$

$$f(x) = \frac{1 + 2\sin(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}.$$

$$\frac{\sin^2(x)}{\cos^2(x)} = \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

Trigonometric identities

Example

Evaluate $I = \int \big[\sec(x) + \tan(x)\big]^2 \, dx$.

Solution: Recall: $f(x) = \big[\sec(x) + \tan(x)\big]^2$,

$$f(x) = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \text{ and } \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx.$

Solution: Recall: $f(x) = \left[\sec(x) + \tan(x)\right]^2,$

$f(x) = \dfrac{1}{\cos^2(x)} + \dfrac{2\sin(x)}{\cos^2(x)} + \dfrac{\sin^2(x)}{\cos^2(x)}$ and $\dfrac{\sin^2(x)}{\cos^2(x)} = \dfrac{1}{\cos^2(x)} - 1.$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx$.

Solution: Recall: $f(x) = \left[\sec(x) + \tan(x)\right]^2$,

$f(x) = \dfrac{1}{\cos^2(x)} + \dfrac{2\sin(x)}{\cos^2(x)} + \dfrac{\sin^2(x)}{\cos^2(x)}$ and $\dfrac{\sin^2(x)}{\cos^2(x)} = \dfrac{1}{\cos^2(x)} - 1.$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

$$I = \int \frac{2\,dx}{\cos^2(x)} + \int \frac{2\sin(x)\,dx}{\cos^2(x)} - \int dx.$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx.$

Solution: Recall: $f(x) = \left[\sec(x) + \tan(x)\right]^2,$

$$f(x) = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \text{ and } \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

$$I = \int \frac{2\,dx}{\cos^2(x)} + \int \frac{2\sin(x)\,dx}{\cos^2(x)} - \int dx. \qquad \begin{aligned} u &= \cos(x), \\ du &= -\sin(x)\,dx. \end{aligned}$$

Trigonometric identities

Example

Evaluate $I = \int \big[\sec(x) + \tan(x)\big]^2 \, dx$.

Solution: Recall: $f(x) = \big[\sec(x) + \tan(x)\big]^2$,

$$f(x) = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \text{ and } \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

$$I = \int \frac{2\,dx}{\cos^2(x)} + \int \frac{2\sin(x)\,dx}{\cos^2(x)} - \int dx. \qquad \begin{aligned} u &= \cos(x), \\ du &= -\sin(x)\,dx. \end{aligned}$$

$$I = 2\tan(x) - 2\int \frac{du}{u^2} - x + c$$

Trigonometric identities

Example

Evaluate $I = \int \left[\sec(x) + \tan(x)\right]^2 dx.$

Solution: Recall: $f(x) = \left[\sec(x) + \tan(x)\right]^2,$

$$f(x) = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \text{ and } \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

$$I = \int \frac{2\,dx}{\cos^2(x)} + \int \frac{2\sin(x)\,dx}{\cos^2(x)} - \int dx. \qquad \begin{aligned} u &= \cos(x), \\ du &= -\sin(x)\,dx. \end{aligned}$$

$$I = 2\tan(x) - 2\int \frac{du}{u^2} - x + c \;\Rightarrow\; I = 2\tan(x) + \frac{2}{\cos(x)} - x + c.$$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- **Polynomial division.**
- Multiplying by 1.

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx.$

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx.$

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$\begin{array}{r|l} & 2x - 3 \\ 2x+3 & 4x^2 \qquad - 7 \\ & -4x^2 - 6x \\ \hline & -6x - 7 \\ & 6x + 9 \\ \hline & 2 \end{array}$$

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$
\begin{array}{r|l}
 & \quad 2x - 3 \\
\hline
2x + 3 & 4x^2 \qquad\quad - 7 \\
 & -4x^2 - 6x \\
\cline{2-2}
 & \qquad -6x - 7 \\
 & \qquad 6x + 9 \\
\cline{2-2}
 & \qquad\qquad 2
\end{array}
\quad \Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.
$$

Polynomial division

Example

Evaluate $I = \displaystyle\int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$
\begin{array}{r}
2x - 3 \\
2x + 3 \overline{\big)\; 4x^2 \qquad - 7} \\
\underline{-\,4x^2 - 6x \quad} \\
-\,6x - 7 \\
\underline{6x + 9} \\
2
\end{array}
\qquad \Rightarrow \qquad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.
$$

$$
I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3}
$$

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$\begin{array}{r|l} & 2x - 3 \\ \hline 2x+3 & 4x^2 \qquad - 7 \\ & -4x^2 - 6x \\ \hline & -6x - 7 \\ & 6x + 9 \\ \hline & 2 \end{array} \quad \Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.$$

$$I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3} \quad \Rightarrow \quad I = x^2 - 3x + \ln(2x + 3) + c.$$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- Polynomial division.
- **Multiplying by 1.**

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{(1+\sin(x))} \frac{(1-\sin(x))}{(1-\sin(x))}\, dx$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1 + \sin(x)}$.

Solution:

$$I = \int \frac{1}{(1 + \sin(x))} \frac{(1 - \sin(x))}{(1 - \sin(x))}\, dx = \int \frac{(1 - \sin(x))}{(1 - \sin^2(x))}\, dx.$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{(1+\sin(x))} \frac{(1-\sin(x))}{(1-\sin(x))}\, dx = \int \frac{(1-\sin(x))}{(1-\sin^2(x))}\, dx.$$

$$I = \int \frac{(1-\sin(x))}{\cos^2(x)}\, dx$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{\big(1+\sin(x)\big)} \frac{\big(1-\sin(x)\big)}{\big(1-\sin(x)\big)}\, dx = \int \frac{\big(1-\sin(x)\big)}{\big(1-\sin^2(x)\big)}\, dx.$$

$$I = \int \frac{\big(1-\sin(x)\big)}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{(1+\sin(x))} \frac{(1-\sin(x))}{(1-\sin(x))}\, dx = \int \frac{(1-\sin(x))}{(1-\sin^2(x))}\, dx.$$

$$I = \int \frac{(1-\sin(x))}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Since $\tan'(x) = \frac{1}{\cos^2(x)}$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1 + \sin(x)}$.

Solution:

$$I = \int \frac{1}{(1 + \sin(x))} \frac{(1 - \sin(x))}{(1 - \sin(x))}\, dx = \int \frac{(1 - \sin(x))}{(1 - \sin^2(x))}\, dx.$$

$$I = \int \frac{(1 - \sin(x))}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Since $\tan'(x) = \frac{1}{\cos^2(x)}$ and $u = \cos(x)$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{\big(1+\sin(x)\big)} \frac{\big(1-\sin(x)\big)}{\big(1-\sin(x)\big)}\,dx = \int \frac{\big(1-\sin(x)\big)}{\big(1-\sin^2(x)\big)}\,dx.$$

$$I = \int \frac{\big(1-\sin(x)\big)}{\cos^2(x)}\,dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\,dx.$$

Since $\tan'(x) = \dfrac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\,dx$,

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1 + \sin(x)}$.

Solution:

$$I = \int \frac{1}{(1 + \sin(x))} \frac{(1 - \sin(x))}{(1 - \sin(x))}\, dx = \int \frac{(1 - \sin(x))}{(1 - \sin^2(x))}\, dx.$$

$$I = \int \frac{(1 - \sin(x))}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Since $\tan'(x) = \frac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\, dx$,

$$I = \tan(x) + \int \frac{du}{u^2}$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{(1+\sin(x))} \frac{(1-\sin(x))}{(1-\sin(x))}\,dx = \int \frac{(1-\sin(x))}{(1-\sin^2(x))}\,dx.$$

$$I = \int \frac{(1-\sin(x))}{\cos^2(x)}\,dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\,dx.$$

Since $\tan'(x) = \dfrac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\,dx$,

$$I = \tan(x) + \int \frac{du}{u^2} = \tan(x) - \frac{1}{u} + c$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1+\sin(x)}$.

Solution:

$$I = \int \frac{1}{\big(1+\sin(x)\big)} \frac{\big(1-\sin(x)\big)}{\big(1-\sin(x)\big)}\, dx = \int \frac{\big(1-\sin(x)\big)}{\big(1-\sin^2(x)\big)}\, dx.$$

$$I = \int \frac{\big(1-\sin(x)\big)}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Since $\tan'(x) = \dfrac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\, dx$,

$$I = \tan(x) + \int \frac{du}{u^2} = \tan(x) - \frac{1}{u} + c = \tan(x) - \frac{1}{\cos(x)} + c.$$

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1 + \sin(x)}$.

Solution:

$$I = \int \frac{1}{(1 + \sin(x))} \frac{(1 - \sin(x))}{(1 - \sin(x))} \, dx = \int \frac{(1 - \sin(x))}{(1 - \sin^2(x))} \, dx.$$

$$I = \int \frac{(1 - \sin(x))}{\cos^2(x)} \, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)} \, dx.$$

Since $\tan'(x) = \frac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\, dx$,

$$I = \tan(x) + \int \frac{du}{u^2} = \tan(x) - \frac{1}{u} + c = \tan(x) - \frac{1}{\cos(x)} + c.$$

We conclude that $I = \tan(x) - \sec(x) + c$. $\triangleleft$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\,dx$.

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\,dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)}$$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}\, dx$$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\,dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}\,dx$$

$$I = \int \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\,dx$$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\,dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}\,dx$$

$$I = \int \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\,dx \quad \text{and} \quad \begin{aligned} \left(\frac{1}{\cos(x)}\right)' &= \frac{\sin(x)}{\cos^2(x)}, \\ \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{1}{\cos^2(x)}. \end{aligned}$$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}\, dx$$

$$I = \int \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\, dx \quad \text{and} \quad \begin{aligned} \left(\frac{1}{\cos(x)}\right)' &= \frac{\sin(x)}{\cos^2(x)}, \\ \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{1}{\cos^2(x)}. \end{aligned}$$

$$I = \int \frac{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)'}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\, dx$$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}\, dx$$

$$I = \int \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\, dx \quad \text{and} \quad \begin{aligned} \left(\frac{1}{\cos(x)}\right)' &= \frac{\sin(x)}{\cos^2(x)}, \\ \left(\frac{\sin(x)}{\cos(x)}\right)' &= \frac{1}{\cos^2(x)}. \end{aligned}$$

$$I = \int \frac{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)'}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)}\, dx \Rightarrow \quad I = \ln\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right) + c.$$