Review for Midterm Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to webwork.
- Midterm Exam 1 covers:
  - Volumes using cross-sections (6.1).
  - Arc-length of curves on the plane (6.3).
  - Work and fluid forces (6.5).
  - The inverse function (7.1).
  - The natural logarithm (7.2).
  - The exponential function (7.3).

Volumes using cross-sections (6.1)

Example

Find the volume of the region obtained by rotation the curve $x(y) = \tan(\pi y/8)$ for $y \in [0, 2]$ about the y-axis.

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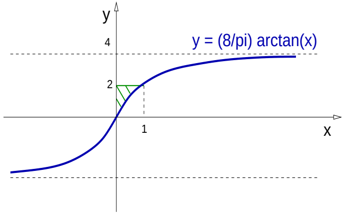

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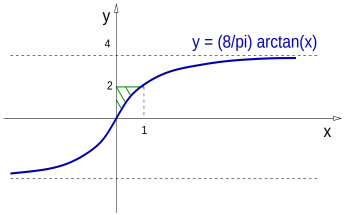

Therefore, $V = \pi \int_0^2 \left[x(y)\right]^2 \, dy$

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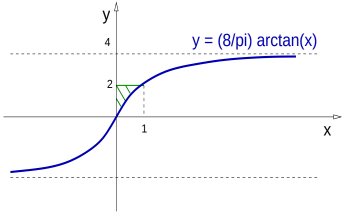

Therefore, $V = \pi \int_0^2 \left[x(y)\right]^2 dy = \pi \int_0^2 \left[\tan\left(\frac{\pi y}{8}\right)\right]^2 dy.$

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Volumes integrating cross-sections: General case.

Example

Find the volume of a pyramid with square base side a and height h.

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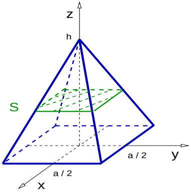

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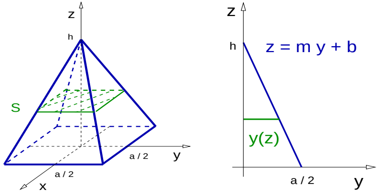

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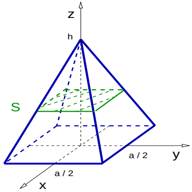

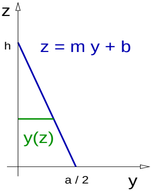

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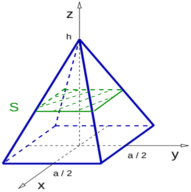

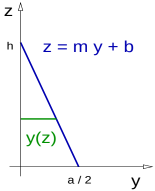

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$$h = z(0) = b, \quad 0 = z(a/2) = m\frac{a}{2} + h \quad \Rightarrow \quad m = -\frac{2h}{a}.$$

$$z(y) = -\frac{2h}{a}\,y + h \quad \Rightarrow \quad y(z) = -\frac{a}{2h}\,(z - h).$$

$$V = \int_0^h \Big[-2\frac{a}{2h}\,(z-h)\Big]^2 dz = \frac{a^2}{h^2}\Big[\frac{(z-h)^3}{3}\Big|_0^h\Big] \Rightarrow V = \frac{1}{3}a^2 h.$$

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Arc-length of curves on the plane (6.3)

Example

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We conclude that $L = 9 - 1/6$. ◁

The main length formula

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The main length formula

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Find the arc-length of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

Solution: Recall: $L = \int_a^b \sqrt{1 + \big[f'(x)\big]^2}\, dx$. We start with

$$f(x) = x^{3/2} \quad \Rightarrow \quad f'(x) = \frac{3}{2}\, x^{1/2} \quad \Rightarrow \quad \big[f'(x)\big]^2 = \frac{9}{4}\, x.$$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx, \quad u = 1 + \frac{9}{4}\, x, \quad du = \frac{9}{4}\, dx.$$

$$L = \int_1^{10} \frac{4}{9} \sqrt{u}\, du = \frac{4}{9}\, \frac{2}{3} \left(u^{3/2} \Big|_1^{10} \right).$$

We conclude that $L = \dfrac{8}{27}(10^{3/2} - 1)$. $\triangleleft$

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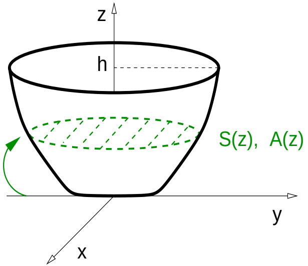
Work and fluid forces: Pumping liquids

Proof: (a) Show: $W = \int_0^{h_1} g\,\delta\,A(z)\,z\,dz$.



The amount of liquid that can be placed at cross-section $S(z)$ is

$$M = \delta\,A(z)\,dz.$$

The force that must be done to lift that amount of liquid is

$$F = g\,[\delta\,A(z)\,dz].$$

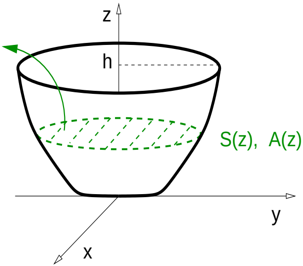
The work done to lift that liquid to height z from $z = 0$ is

$$W(z) = z\,g\,[\delta\,A(z)\,dz].$$

The work to fill in the container up to h_1 is $W = \int_0^{h_1} g\,\delta\,A(z)\,z\,dz$.

Work and fluid forces: Pumping liquids

Proof: (b) Show: $W = \int_0^{h_1} g\,\delta\,A(z)\,(h-z)\,dz.$



The force that must be done to lift the liquid in $S(z)$ is

$$F = g\,[\delta\,A(z)\,dz].$$

The work done to lift that liquid from a height z to h is

$$W(z) = (h-z)\,g\,[\delta\,A(z)\,dz].$$

The work to empty the container initially filled up to h_1 is

$$W = \int_0^{h_1} g\,\delta\,A(z)\,(h-z)\,dz.$$

Work and fluid forces: Pumping liquids

Example

A rectangular container with sides a, b, and height h, is filled with water. Find the work needed to empty the container if the water is pumped from the top of the tank. Recall the water density is $\delta = 1000\ Kg/m^3$, and the gravity acceleration is $g = 10\ m/s^2$.

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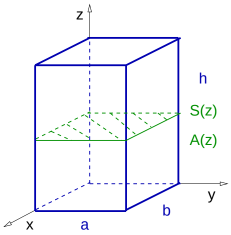

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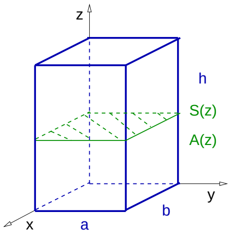

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$$F = g\,[\delta\, A(z)\, dz] = g\delta(ab)\, dz$$

The work done to lift that liquid from a height z to h is

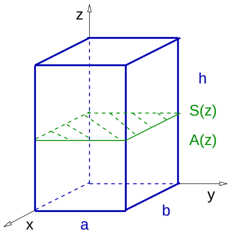
$$W(z) = g\delta(ab)(h - z)\, dz.$$

Work and fluid forces: Pumping liquids

Example

A rectangular container with sides a, b, and height h, is filled with water. Find the work needed to empty the container if the water is pumped from the top of the tank. Recall the water density is $\delta = 1000\ Kg/m^3$, and the gravity acceleration is $g = 10\ m/s^2$.

Solution:



The force is the water weight:

$$F = g\,[\delta\, A(z)\, dz] = g\delta(ab)\, dz$$

The work done to lift that liquid from a height z to h is

$$W(z) = g\delta(ab)(h - z)\, dz.$$

To empty the container: $W = g\delta(ab) \int_0^h (h - z)\, dz$

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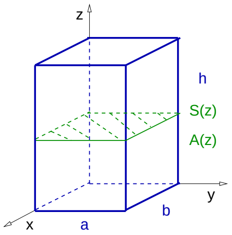

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Work and fluid forces: Springs

Remark: The force of a spring, $F(x) = k\,x$ is called *Hooke's Law*.

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Find the minimum work needed to compress a spring with constant $k = 3\ N/m$ a distance of $d\ m$ from the spring rest position.

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Solution: From Hooke's Law we know that $60\ N = k\ (3)\ m$, that is, $k = 20\ N/m$. The previous problem implies $W = kd^2/2$, that is,

$$W = 20\ \frac{N}{m}\ \frac{4^2}{2}\ m^2 \quad \Rightarrow \quad W = 160\ J. \qquad \lhd$$

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  - **The inverse function (7.1).**
  - The natural logarithm (7.2).
  - The exponential function (7.3).

The inverse function (7.1).

Example

Find the inverse of $f(x) = 8(x-2)^2 + 3$ for $x \geqslant 2$.

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We obtain $(f^{-1})'(8) = \dfrac{1}{12}$. ◁

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The natural logarithm (7.2)

Example

Simplify $f(x) = \ln\left(\frac{\sin^5(2t)}{7}\right)$, and find the derivatives of $g(x) = 3\ln(6\ln(x))$, and $h(x) = \ln\left(\sqrt{25\sin(x)\cos(x)}\right)$.

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The exponential function (7.3)

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Solve for y in terms of x the equation

$$\ln(3y - 5) + \ln(2) = 4x + \ln(2x).$$

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$$\ln(3y-5)+\ln(2)=4x+\ln(2x).$$

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$$y(x) = \int 5\,e^{5x}\,\sin\big(e^{5x} - 2\big)\,dx + c$$

The exponential function (7.3)

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Solve the initial value problem

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We conclude that $c = 1$,

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We conclude that $c = 1$, so $y(x) = -\cos\big(e^{5x} - 2\big) + 1$. $\lhd$

Solving differential equations (Sect. 7.4)

Today: Applications.

- Review: Overview of differential equations.
- Population growth.
- Radioactive decay.
- Newton's Law of Cooling.

Previous class:

- Overview of differential equations.
- Exponential growth.
- Separable differential equations.

Review: Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

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Review: Overview of differential equations.

Example

Find all solutions y to the equation $y'(x) = \dfrac{e^{2x-y}}{e^{x+y}}$.

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We conclude that $y(x) = \dfrac{1}{2}\ln\big(2\,(e^x + c)\big)$. $\triangleleft$

Solving differential equations (Sect. 7.4)

Today: Applications.

- Review: Overview of differential equations.
- **Population growth.**
- Radioactive decay.
- Newton's Law of Cooling.

Population growth

Example

Assume the world population growth is described by $y(t) = y_0\, e^{k(t-t_0)}$, with t measured in years.

(a) If in $1960 - 1961$ the population increased by 2%, find k.

(b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above.

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Solution: (a) $y(1961) = \left(1 + \dfrac{2}{100}\right) y(1960)$,

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$$e^k = 1.02 \;\Rightarrow\; k = \ln(1.02)$$

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$$e^{k} = 1.02 \;\Rightarrow\; k = \ln(1.02) \simeq 0.02.$$

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$$y_0\, e^{k(1961-t_0)} = \frac{102}{100}\, y_0\, e^{k(1960-t_0)}$$

$$e^{k1961}\, e^{-kt_0} = 1.02\, e^{k1960}\, e^{-kt_0} \;\Rightarrow\; e^{k(1961-1960)} = 1.02.$$

$e^{k} = 1.02 \;\Rightarrow\; k = \ln(1.02) \simeq 0.02.$ Hence $y(t) = y_0\, e^{(0.02)(t-t_0)}$.

Population growth

Example

Assume the world population growth is described by $y(t) = y_0\, e^{k(t-t_0)}$, with t measured in years.

(a) If in $1960 - 1961$ the population increased by 2%, find k.

(b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above.

Solution: Recall: $y(t) = y_0\, e^{(0.02)(t-t_0)}$.

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We only need to evaluate $y(2012) = 3\, e^{(0.02)52} = 8.5$ billions. $\triangleleft$

Solving differential equations (Sect. 7.4)

Today: Applications.

- Review: Overview of differential equations.
- Population growth.
- **Radioactive decay.**
- Newton's Law of Cooling.

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Recall $\tau = \ln(2)/k$

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We obtain $t_1 = [5730/\ln(2)]\, \ln\Big(\dfrac{100}{14}\Big)$, hence $t_1 = 16,253$ years. $\triangleleft$

Solving differential equations (Sect. 7.4)

Today: Applications.

- Review: Overview of differential equations.
- Population growth.
- Radioactive decay.
- **Newton's Law of Cooling.**

Newton's Law of Cooling.

Remarks:

- The temperature difference $\Delta T = T - T_0$ between the temperature of an object, T, and the constant temperature of the surrounding medium where it is placed, T_s, evolves in time t following the equation

$$(\Delta T)' = -k(\Delta T), \quad T(0) = T_0, \quad k > 0.$$

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- The constant k depends on the material and the surroundings.

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A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C? while

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$$20 = T(2) = 40\, e^{-2k} \;\Rightarrow\; \ln(1/2) = -2k \;\Rightarrow\; k = \frac{1}{2}\ln(2).$$

$$T(t) = 40\, e^{-t\ln(\sqrt{2})} + 5$$

Newton's Law of Cooling.

Example

A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C? while

Solution: We know that $T(t) = (T_0 - T_s)\, e^{-kt} + T_s$, and also

$$T_0 = 45, \qquad T_s = 5, \qquad T(2) = 25.$$

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Inverse trigonometric functions (Sect. 7.6)

Today: Definitions and properties.

- Domains restrictions and inverse trigs.
- Evaluating inverse trigs at simple values.
- Few identities for inverse trigs.

Next class: Derivatives and integrals.

- Derivatives.
- Anti-derivatives.
- Usual substitutions

Domains restrictions and inverse trigs

Remark: The trigonometric functions defined on their biggest domain are not invertible.

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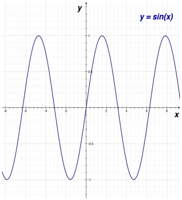

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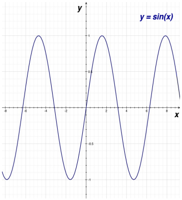

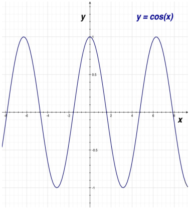

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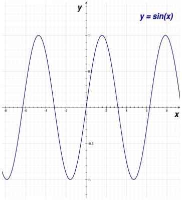

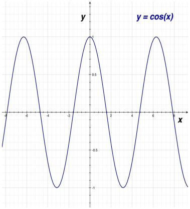

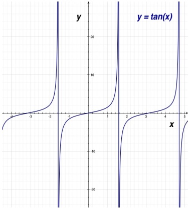

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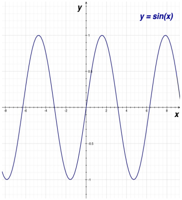

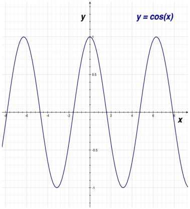

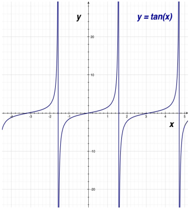

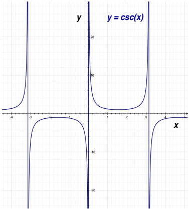

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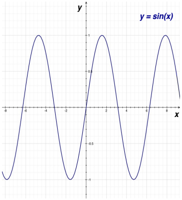

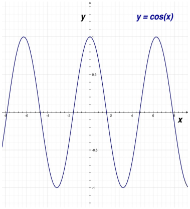

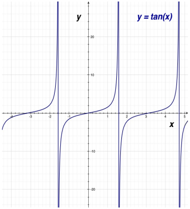

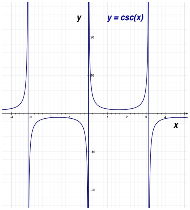

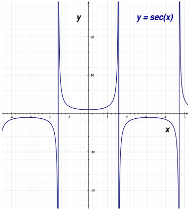

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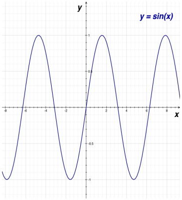

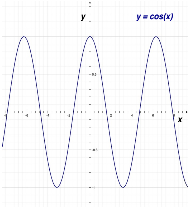

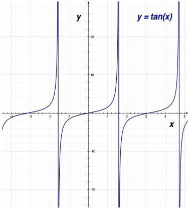

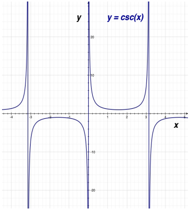

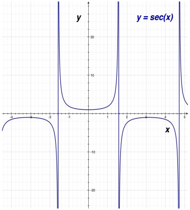

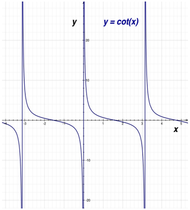

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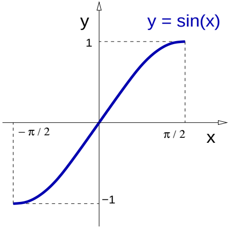

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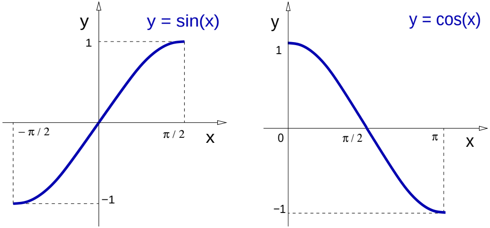

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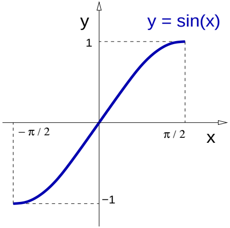

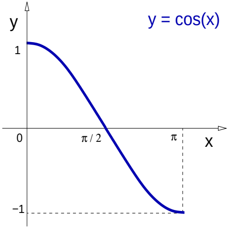

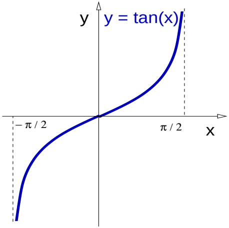

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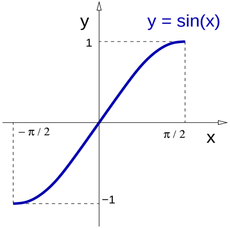

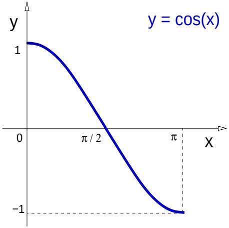

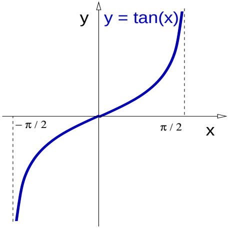

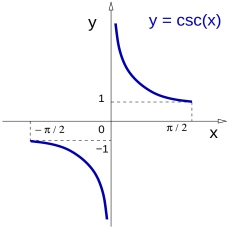

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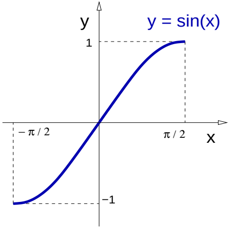

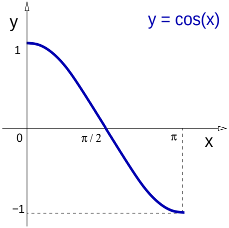

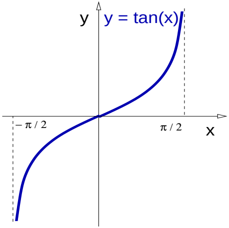

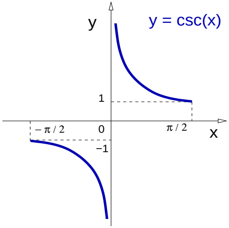

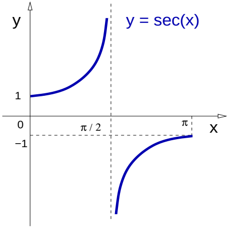

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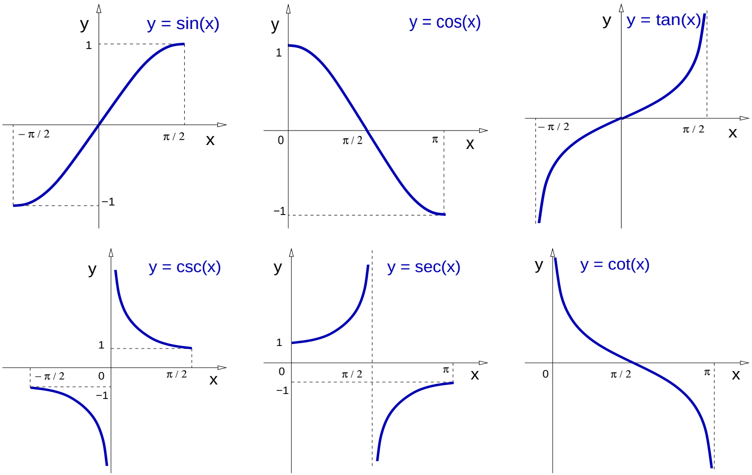

Domains restrictions and inverse trigs

Definition (arc{trig} is the inverse of {trig})

- The function $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ is the inverse of $\sin : [-\pi/2, \pi/2] \to [-1,1]$.
- The function $\arccos : [-1,1] \to [0,\pi]$ is the inverse of $\cos : [0,\pi] \to [-1,1]$.
- The function $\arctan : (-\infty,\infty) \to [-\pi/2, \pi/2]$ is the inverse of $\tan : [-\pi/2, \pi/2] \to (-\infty,\infty)$.
- The function $\operatorname{arccsc} : (-\infty,-1] \cup [1,\infty) \to [-\pi/2, \pi/2] - \{0\}$ is the inverse of $\csc : [-\pi/2, \pi/2] - \{0\} \to (-\infty,-1] \cup [1,\infty)$.
- The function $\operatorname{arcsec} : (-\infty,-1] \cup [1,\infty) \to [0,\pi] - \{\pi/2\}$ is the inverse of $\sec : [0,\pi] - \{\pi/2\} \to (-\infty,-1] \cup [1,\infty)$.
- The function $\operatorname{arccot} : (-\infty,\infty) \to [0,\pi]$ is the inverse of $\cot : [0,\pi] \to (-\infty,\infty)$.

Domains restrictions and inverse trigs

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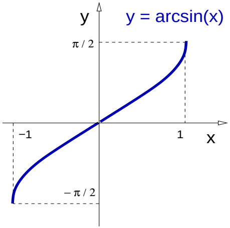

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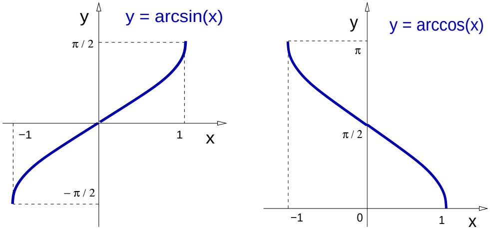

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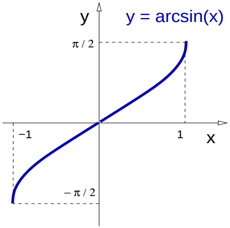

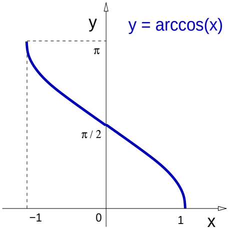

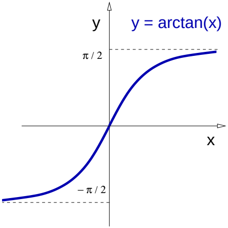

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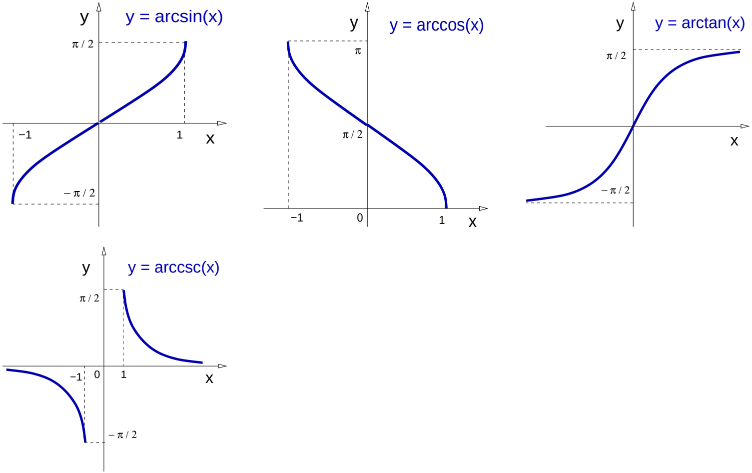

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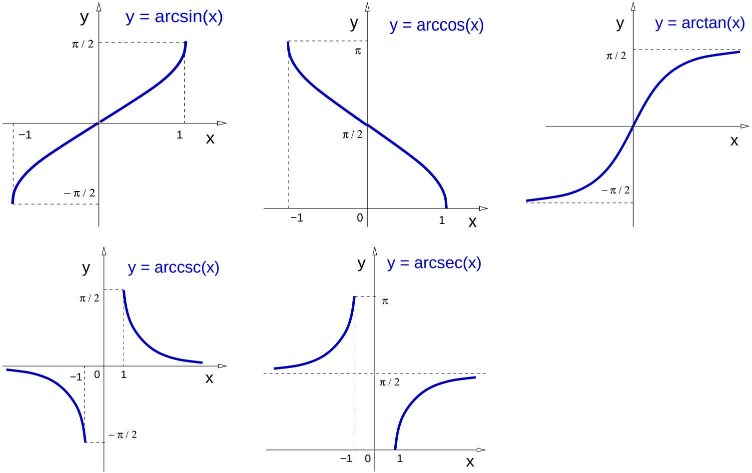

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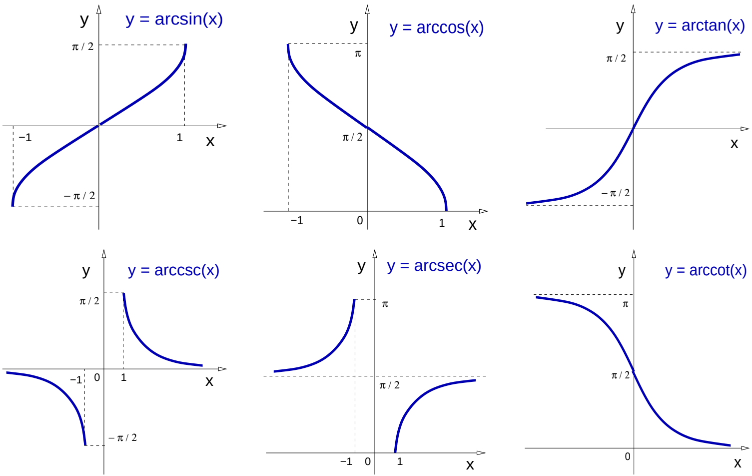

Inverse trigonometric functions (Sect. 7.6)

Today: Definitions and properties.

- Domains restrictions and inverse trigs.
- **Evaluating inverse trigs at simple values.**
- Few identities for inverse trigs.

Evaluating inverse trigs at simple values

Notation: In the literature is common the notation $\sin^{-1} = \arcsin$, and similar for the rest of the trigonometric functions.

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Remark: sin, cos have simple values at particular angles.

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θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Evaluating inverse trigs at simple values

Remark: the symmetry properties of the sine and cosine can be used to evaluate them at a bigger set of angles.

$$\sin(-x) = -\sin(x), \qquad \sin(\pi - x) = \sin(x).$$

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$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos(\theta).$$

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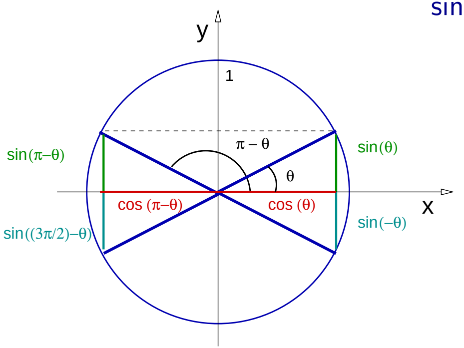

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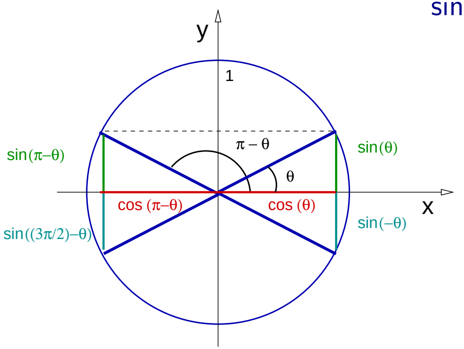

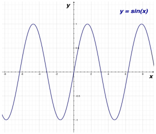

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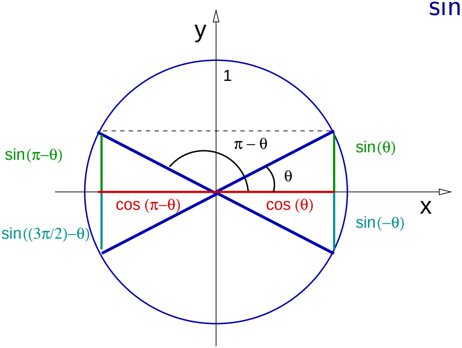

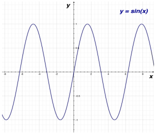

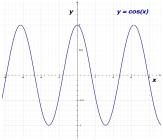

Evaluating inverse trigs at simple values

Example

Find the values: $\tan(\pi/3)$, $\sec(2\pi/3)$, $\csc(-\pi/6)$.

Evaluating inverse trigs at simple values

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Solution:

$$\tan\left(\frac{\pi}{3}\right)$$

Evaluating inverse trigs at simple values

Example

Find the values: $\tan(\pi/3)$, $\sec(2\pi/3)$, $\csc(-\pi/6)$.

Solution:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})}$$

Evaluating inverse trigs at simple values

Example

Find the values: $\tan(\pi/3)$, $\sec(2\pi/3)$, $\csc(-\pi/6)$.

Solution:

$$\tan\left(\frac{\pi}{3}\right) = \frac{\sin(\frac{\pi}{3})}{\cos(\frac{\pi}{3})} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$

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Evaluating inverse trigs at simple values

Example

Find the values: $\arcsin(\sqrt{3}/2)$, $\arccos(1/\sqrt{2})$.

Evaluating inverse trigs at simple values

Example

Find the values: $\arcsin(\sqrt{3}/2)$, $\arccos(1/\sqrt{2})$.

Solution:

$$\arcsin\left(\frac{\sqrt{3}}{2}\right) = \theta \quad \Leftrightarrow \quad \sin(\theta) = \frac{\sqrt{3}}{2},$$

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We conclude that $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

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We conclude that $\arcsin\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$.

$$\arccos\left(\frac{1}{\sqrt{2}}\right) = \theta \quad \Leftrightarrow \quad \cos(\theta) = \frac{1}{\sqrt{2}}$$

Evaluating inverse trigs at simple values

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Find the values: $\arcsin(\sqrt{3}/2)$, $\arccos(1/\sqrt{2})$.

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Evaluating inverse trigs at simple values

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We conclude that $\arccos\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$. $\triangleleft$

Evaluating inverse trigs at simple values

Example

Given that $x = \text{arcsec}(-5\sqrt{5})$, find $\sin(x)$.

Evaluating inverse trigs at simple values

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Given that $x = \operatorname{arcsec}(-5\sqrt{5})$, find $\sin(x)$.

Solution: Recall:

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How do we find $\sin(x)$? Use trigonometric identities:

$$\sin(x) = \pm\sqrt{1 - \cos^2(x)} = \pm\sqrt{1 - \frac{1}{125}}$$

Evaluating inverse trigs at simple values

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How do we find $\sin(x)$? Use trigonometric identities:

$$\sin(x) = \pm\sqrt{1 - \cos^2(x)} = \pm\sqrt{1 - \frac{1}{125}} \;\Rightarrow\; \sin(x) = \pm\sqrt{\frac{124}{125}}.$$

◁

Inverse trigonometric functions (Sect. 7.6)

Today: Definitions and properties.

- Domains restrictions and inverse trigs.
- Evaluating inverse trigs at simple values.
- **Few identities for inverse trigs.**

Few identities for inverse trigs

Theorem

For all $x \in [-1, 1]$ *the following identities hold,*

$$\arccos(x) + \arccos(-x) = \pi, \qquad \arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

Few identities for inverse trigs

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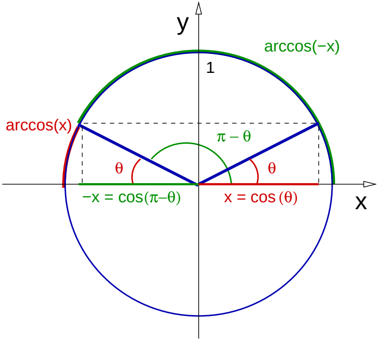

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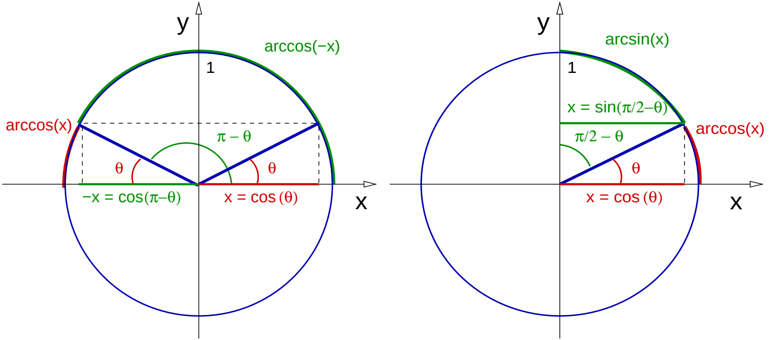

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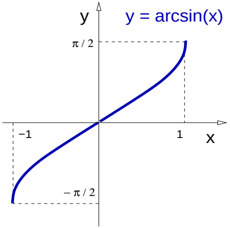

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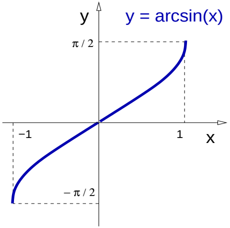

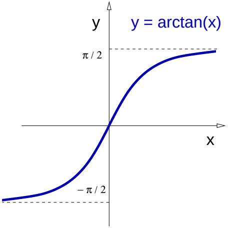

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