The exponential function (Sect. 7.3)

- The inverse of the logarithm.
- Derivatives and integrals.
- Algebraic properties.

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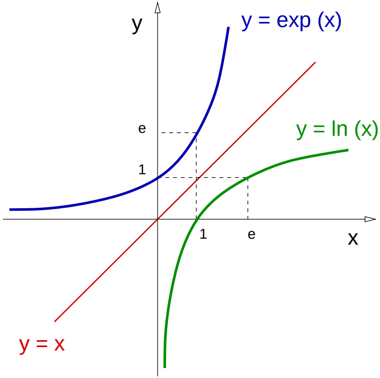

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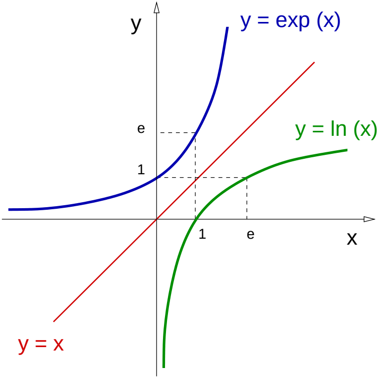

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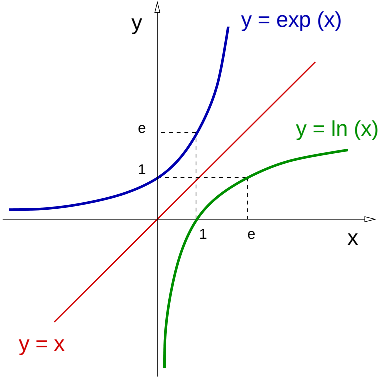

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Find x solution of $e^{3x+1} = 2$.

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Remark: In particular: $(e^{ax})' = a\, e^{ax}$, for $a \in \mathbb{R}$.

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Since, $I = \frac{1}{6}(e^1 - e^0)$, we obtain $I = \frac{1}{6}(e - 1)$. $\triangleleft$

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Solution: Recall that $\left(e^{2x^2}\right)'$

Derivatives and integrals

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Derivatives and integrals

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◁

Derivatives and integrals

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Find the solution to the initial value problem

$$y''(x) = 18\, e^{3x}, \quad y(0) = 1, \quad y'(0) = 2$$

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We now need to integrate one more time.

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We conclude that $y(x) = 2\, e^{3x} - 4x - 1.$ $\lhd$

The exponential function (Sect. 7.3)

- The inverse of the logarithm.
- Derivatives and integrals.
- **Algebraic properties.**

Algebraic properties

Remark: The algebraic properties on natural logarithms translate into algebraic properties of the exponential function.

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Theorem

For every a, b, c $\in \mathbb{R}$, and every rational number, q, hold

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We conclude that $e^{a+b} = e^a\, e^b$. □

Algebraic properties

Example

Simplify the expression $\left(\frac{e^{x-\ln(2)}}{e}\right)^3$.

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Solution:

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Simplify the expression $\left(\frac{e^{x-\ln(2)}}{e}\right)^3$.

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We conclude that $\left(\frac{e^{x-\ln(2)}}{e}\right)^3 = \frac{1}{8}\, e^{3(x-1)}$. $\triangleleft$

The exponential function (Sect. 7.3)

- Review: The exponential function e^x.
- Computing the number e.
- The exponential function a^x.
- Derivatives and integrals.
- Logarithms with base $a \in \mathbb{R}$.

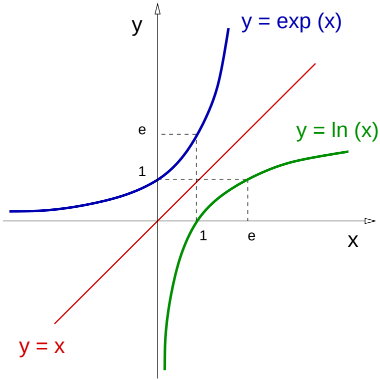
Review: The exponential function e^x

Definition

The *exponential function*, $\exp : \mathbb{R} \to (0, \infty)$, is the inverse of the natural logarithm, that is,

$$\exp(x) = y \Leftrightarrow x = \ln(y).$$

Notation: $\exp(x) = e^x$.



Remark: Since $\ln(1) = 0$, then $e^0 = 1$.
Since $\ln(e) = 1$, then $e^1 = e$.

$$\left(e^{ax}\right)' = a\, e^{ax}, \qquad \int e^{ax}\, dx = \frac{e^{ax}}{a} + c.$$

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(d) $\big(e^a\big)^q = e^{qa}$.

Proof: Only of (a):

$$\ln\big(e^{a+b}\big) = a + b = \ln\big(e^a\big) + \ln\big(e^b\big) = \ln\big(e^a\, e^b\big).$$

We conclude that $e^{a+b} = e^a\, e^b$. □

Algebraic properties

Example

Simplify the expression $\left(\frac{e^{x-\ln(2)}}{e}\right)^3$.

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We conclude that $\left(\dfrac{e^{x-\ln(2)}}{e}\right)^3 = \dfrac{1}{8}\, e^{3(x-1)}$. ◁

The exponential function (Sect. 7.3)

- Review: The exponential function e^x.
- **Computing the number e.**
- The exponential function a^x.
- Derivatives and integrals.
- Logarithms with base $a \in \mathbb{R}$.

Computing the number e.

Theorem

The number e defined as $\ln(e) = 1$ *can be obtained as*

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

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Therefore, $\ln\big[\lim_{h\to 0}(1+h)^{1/h}\big] = 1$. But ln is a one-to-one function,

and $\ln(e) = 1$, hence $e = \lim_{h\to 0}(1+h)^{1/h}$. □

Computing the number e.

Remark: The convergence in $e = \lim_{h \to 0} (1 + h)^{1/h}$ is slow.

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Remark: The convergence in $e = \lim_{h \to 0} (1 + h)^{1/h}$ is slow.

- For $h = 1$, $e_h = 2$.
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Remark: $e = 2.71828182\ldots$.

The exponential function (Sect. 7.3)

- Review: The exponential function e^x.
- Computing the number e.
- **The exponential function a^x.**
- Derivatives and integrals.
- Logarithms with base $a \in \mathbb{R}$.

The exponential function a^x

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The exponentiation function on base $a \in (0, \infty)$ is the function $\exp[a] : \mathbb{R} \to (0, \infty)$ given by

$$\exp[a](x) = e^{x \ln(a)}.$$

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- Also $\exp[a](m/n) = e^{(m/n)\ln(a)} = e^{\ln(a^{m/n})} = a^{m/n}$.
- Notation: $\exp[a](x) = a^x$, for $x \in \mathbb{R}$.

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Remark: The algebraic properties of e^x also hold for a^x.

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Theorem

For every $a \in (0, \infty)$, b, $c \in \mathbb{R}$, and every rational number, q, hold

(a) $a^{b+c} = a^b, a^c$;

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$$a^{(b+c)} = e^{(b+c)\ln(a)}$$

The exponential function a^x

Remark: The algebraic properties of e^x also hold for a^x.

Theorem

For every $a \in (0, \infty)$, b, $c \in \mathbb{R}$, and every rational number, q, hold

(a) $a^{b+c} = a^b, a^c$;

(b) $a^{-b} = \dfrac{1}{a^b}$;

(c) $a^{b-c} = \dfrac{a^b}{a^c}$;

(d) $\left(a^a\right)^q = a^{qa}$.

Proof: Only of (a):

$$a^{(b+c)} = e^{(b+c)\ln(a)} = e^{b\ln(a)+c\ln(a)}$$

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$$a^{(b+c)} = e^{(b+c)\ln(a)} = e^{b\ln(a)+c\ln(a)} = e^{b\ln(a)}\, e^{c\ln(a)}.$$

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$$a^{(b+c)} = e^{(b+c)\ln(a)} = e^{b\ln(a)+c\ln(a)} = e^{b\ln(a)}\, e^{c\ln(a)}.$$

We conclude that $a^{(b+c)} = a^b\, a^c$. $\square$

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}}$$

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}} = e^{(\pi+\sqrt{2})\ln(3)}$$

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}} = e^{(\pi+\sqrt{2})\ln(3)} = e^{(3.14...+1.41...)(1.099...)}$$

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}} = e^{(\pi+\sqrt{2})\ln(3)} = e^{(3.14...+1.41...)(1.099...)} = 149.167...$$

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Example

Compute $2^{-\pi}$.

The exponential function a^x

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The exponential function a^x

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Example

Compute $2^{-\pi}$.

Solution:

$$2^{-\pi} = \frac{1}{2^{\pi}} = \frac{1}{e^{\pi\ln(2)}}$$

The exponential function a^x

Example

Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}} = e^{(\pi+\sqrt{2})\ln(3)} = e^{(3.14...+1.41...)(1.099...)} = 149.167...$$

◁

Example

Compute $2^{-\pi}$.

Solution:

$$2^{-\pi} = \frac{1}{2^{\pi}} = \frac{1}{e^{\pi\ln(2)}} = \frac{1}{8.825...}$$

◁

The exponential function (Sect. 7.3)

- Review: The exponential function e^x.
- Computing the number e.
- The exponential function a^x.
- **Derivatives and integrals.**
- Logarithms with base $a \in \mathbb{R}$.

Derivatives and integrals

Theorem

For every $a \in (0, \infty)$, $c \in \mathbb{R}$, *and differentiable function* u *holds,*

$$(a^x)' = \ln(a)\, a^x, \qquad (a^u)' = \ln(a)\, a^u\, u'.$$

In addition, if $a \neq 1$, *then*

$$\int a^x\, dx = \frac{a^x}{\ln(a)} + c.$$

Derivatives and integrals

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Proof of the first equation:

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$$(a^x)' = \left(e^{x \ln(a)}\right)'$$

Derivatives and integrals

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For every $a \in (0, \infty)$, $c \in \mathbb{R}$, *and differentiable function* u *holds,*

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Proof of the first equation:

$$(a^x)' = \left(e^{x \ln(a)}\right)' = \ln(a) \left(e^{x \ln(a)}\right),$$

Derivatives and integrals

Theorem

For every $a \in (0,\infty)$, $c \in \mathbb{R}$, and differentiable function u holds,

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In addition, if $a \neq 1$, then

$$\int a^x\, dx = \frac{a^x}{\ln(a)} + c.$$

Proof of the first equation:

$$(a^x)' = \left(e^{x\ln(a)}\right)' = \ln(a)\left(e^{x\ln(a)}\right),$$

that is, $(a^x)' = \ln(a)\, a^x$. $\square$

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

Solution: The derivative is $\left(5^x\right)'$

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

Solution: The derivative is $\left(5^x\right)' = \ln(5)\, 5^x$.

Derivatives and integrals

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Compute both the derivative and a primitive of $f(x) = 5^x$.

Solution: The derivative is $\left(5^x\right)' = \ln(5)\, 5^x$.

The antiderivatives are $\int 5^x \, dx$

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

Solution: The derivative is $\big(5^x\big)' = \ln(5)\,5^x$.

The antiderivatives are $\displaystyle\int 5^x\,dx = \frac{1}{\ln(5)}\,5^x + c$, for $c \in \mathbb{R}$. $\triangleleft$

Derivatives and integrals

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Compute both the derivative and a primitive of $f(x) = 5^{3x}$.

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Derivatives and integrals

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Compute both the derivative and a primitive of $f(x) = 5^x$.

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The antiderivatives are $\displaystyle\int 5^x\,dx = \frac{1}{\ln(5)}\,5^x + c$, for $c \in \mathbb{R}$. $\lhd$

Example

Compute both the derivative and a primitive of $f(x) = 5^{3x}$.

Solution: $\left(5^{3x}\right)' = \ln(5)\,5^{3x}(3x)'$, hence $\left(5^{3x}\right)' = 3\ln(5)\,5^{3x}$,

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

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For the antiderivatives use $u = 3x$,

Derivatives and integrals

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Derivatives and integrals

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$$I = \int 5^{3x}\, dx$$

Derivatives and integrals

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Derivatives and integrals

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Derivatives and integrals

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For the antiderivatives use $u = 3x$, $du = 3\, dx$,

$$I = \int 5^{3x}\, dx = \int 5^u \frac{du}{3} = \frac{1}{3}\frac{5^u}{\ln(5)} \quad \Rightarrow \quad I = \frac{5^{3x}}{3\ln(5)} + c.$$

$\triangleleft$

Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx$.

Derivatives and integrals

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Derivatives and integrals

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Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx.$

Solution: Use the substitution $u = \sin(x)$, then $du = \cos(x)\, dx$.

Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx$.

Solution: Use the substitution $u = \sin(x)$, then $du = \cos(x)\, dx$.

$$I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx$$

Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx$.

Solution: Use the substitution $u = \sin(x)$, then $du = \cos(x)\, dx$.

$$I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx = \int \left(\frac{1}{7}\right)^{u} du$$

Derivatives and integrals

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Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\, dx.$

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Now substitute back,

$$I = -\frac{1}{\ln(7)} \left(\frac{1}{7}\right)^{\sin(x)} + c. \qquad \triangleleft$$

The exponential function (Sect. 7.3)

- Review: The exponential function e^x.
- Computing the number e.
- The exponential function a^x.
- Derivatives and integrals.
- **Logarithms with base $a \in \mathbb{R}$.**

Logarithms with base $a \in \mathbb{R}$.

Remarks:

- The function $a^x = e^{x \ln(a)}$ is one-to-one, so invertible.

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- $\log_a(x)$, a logarithm with base a, is the inverse of a^x.

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- The function $a^x = e^{x \ln(a)}$ is one-to-one, so invertible.
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- The function $\log_a$ is proportional to ln.

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Definition

For every positive a with $a \neq 1$ the function $\log_a : (0, \infty) \to \mathbb{R}$ is given by

$$\log_a(x) = y \quad \Leftrightarrow \quad x = a^y.$$

Logarithms with base $a \in \mathbb{R}$.

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Theorem

For positive a with $a \neq 1$ holds $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Logarithms with base $a \in \mathbb{R}$.

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Proof: $\log_a(x) = y \Leftrightarrow x = a^y$

Logarithms with base $a \in \mathbb{R}$.

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For positive a with $a \neq 1$ *holds* $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof: $\log_a(x) = y \Leftrightarrow x = a^y = e^{y \ln(a)} \Leftrightarrow \ln(x) = y \ln(a)$.

Therefore, $\ln(x) = \log_a(x) \ln(a)$

Logarithms with base $a \in \mathbb{R}$.

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For positive a with $a \neq 1$ holds $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof: $\log_a(x) = y \Leftrightarrow x = a^y = e^{y \ln(a)} \Leftrightarrow \ln(x) = y \ln(a)$.

Therefore, $\ln(x) = \log_a(x) \ln(a) \quad \Rightarrow \quad \log_a(x) = \dfrac{\ln(x)}{\ln(a)}$. □

Logarithms with base $a \in \mathbb{R}$.

Theorem

For every positive a, $a \neq 1$, and differentiable function u holds,

$$\log_a'(x) = \frac{1}{\ln(a)\,x}, \qquad \big[\log_a(u)\big]' = \frac{u'}{\ln(a)\,u}.$$

Logarithms with base $a \in \mathbb{R}$.

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For every positive a, $a \neq 1$, and differentiable function u holds,

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Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$,

Logarithms with base $a \in \mathbb{R}$.

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For every positive a, $a \neq 1$, and differentiable function u holds,

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Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, then

$$\log_a'(x) = \frac{1}{\ln(a)}\,\frac{1}{x}$$

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For every positive a, $a \neq 1$, and differentiable function u holds,

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Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, then

$$\log_a'(x) = \frac{1}{\ln(a)} \frac{1}{x}$$

□

Example

Compute the derivative of $f(x) = \log_2(3x^3 + 2)$.

Logarithms with base $a \in \mathbb{R}$.

Theorem

For every positive a, $a \neq 1$, and differentiable function u holds,

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□

Example

Compute the derivative of $f(x) = \log_2(3x^3 + 2)$.

Solution: $f'(x) = \dfrac{1}{\ln(2)}\,\ln'(3x^2 + 2)$

Logarithms with base $a \in \mathbb{R}$.

Theorem

For every positive a, $a \neq 1$, and differentiable function u holds,

$$\log_a'(x) = \frac{1}{\ln(a)\, x}, \qquad \big[\log_a(u)\big]' = \frac{u'}{\ln(a)\, u}.$$

Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, then

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Compute the derivative of $f(x) = \log_2(3x^3 + 2)$.

Solution: $f'(x) = \dfrac{1}{\ln(2)}\, \ln'(3x^2 + 2) = \dfrac{1}{\ln(2)}\,\dfrac{1}{(3x^2+2)}\, 6x.$

Logarithms with base $a \in \mathbb{R}$.

Theorem

For every positive a, $a \neq 1$, and differentiable function u holds,

$$\log_a'(x) = \frac{1}{\ln(a)\, x}, \qquad \big[\log_a(u)\big]' = \frac{u'}{\ln(a)\, u}.$$

Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, then

$$\log_a'(x) = \frac{1}{\ln(a)}\,\frac{1}{x}$$

$\square$

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We conclude: $f'(x) = \dfrac{6x}{\ln(2)\,(3x^2 + 2)}.$ $\triangleleft$

Solving differential equations (Sect. 7.4)

- Overview of differential equations.
- Exponential growth.
- Separable differential equations.

Next class: Applications.

- Population growth
- Radioactive decay.
- Heat transfer.

Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

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with m the particle mass and f the force acting on the particle.

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The following are examples of differential equations:

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Overview of differential equations.

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Verify that the functions $y(x) = c\,e^{-2x} - \dfrac{3}{2}$, for every $c \in \mathbb{R}$, are solutions to the differential equation $y' = 2y + 3$.

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Therefore, $y' = 2y + 3$ for all $c \in \mathbb{R}$. $\triangleleft$

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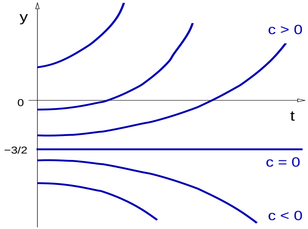

Solving differential equations (Sect. 7.4)

- Overview of differential equations.
- **Exponential growth.**
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Exponential growth

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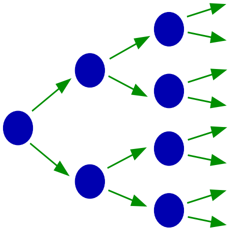

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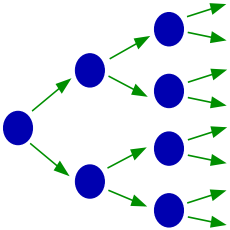

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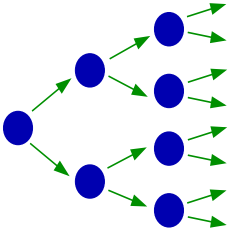

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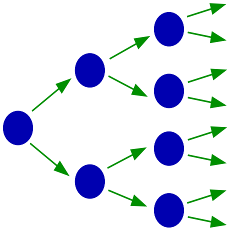

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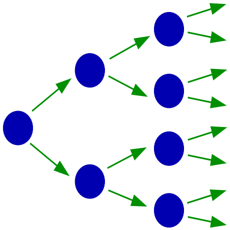

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$$\int \frac{du}{u} = -3 \int dx \quad \Rightarrow \quad \ln(|u|) = -3x + c.$$

Substitute back $y(x) = u$, and exponentiate both sides,

$$|y(x)| = e^{-3x+c} = e^{-3x}\,e^{c}$$

Exponential growth

Example

Find every positive function y solution of the differential equation

$$\frac{dy}{dx}(x) = -3\, y(x).$$

Solution: This differential equation is particularly simple to solve.

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Solving differential equations (Sect. 7.4)

- Overview of differential equations.
- Exponential growth.
- **Separable differential equations.**

Separable differential equations

Definition

Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a differential equation on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *separable* iff the equation has the form

$$h(y)\, y'(x) = g(x).$$

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Determine whether the differential equation below is separable,

$$y'(x) = \frac{x^2}{1 - y^2(x)}.$$

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Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(x) = c\, x^2, \quad h(y) = c\,(1 - y^2), \quad c \in \mathbb{R}.$$

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◁

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Indeed, $1 = y(0) = \dfrac{2}{0 - 2c}$, so $1 = -\dfrac{1}{c}$, hence $c = -1$.

We conclude that $y(t) = \dfrac{2}{\sin(2t) + 2}$. ◁