Natural Logarithms (Sect. 7.2)

- Definition as an integral.
- The derivative and properties.
- The graph of the natural logarithm.
- Integrals involving logarithms.
- Logarithmic differentiation.

Definition as an integral

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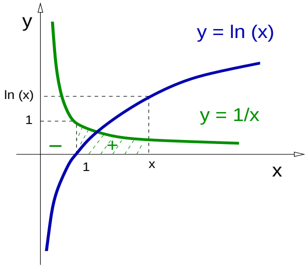

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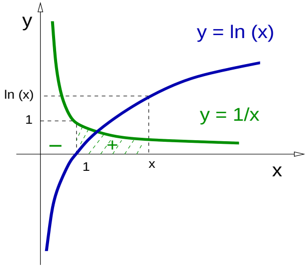

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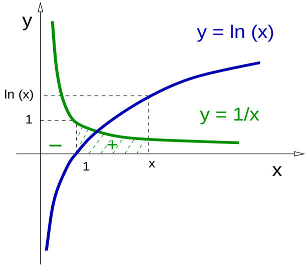

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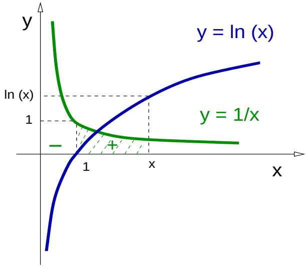

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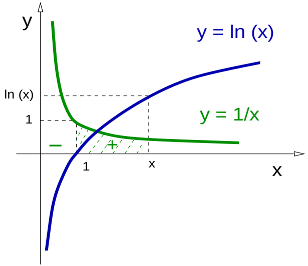

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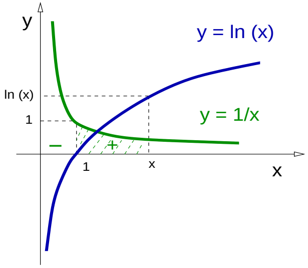

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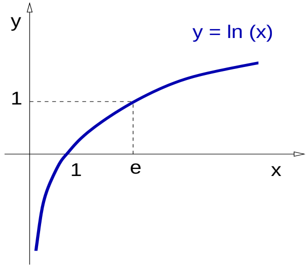

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Theorem (Derivative of ln)

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Example

Find the derivative of $y(x) = \ln(3x)$, and $z(x) = \ln\big(2x^2 + \cos(x)\big)$.

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Remark: $y(x) = \ln(3x)$, satisfies $y'(x) = \ln'(x)$.

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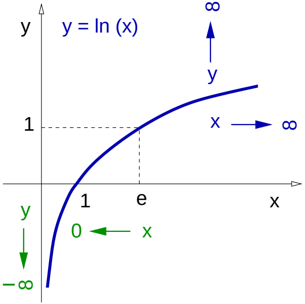
Remarks:

The graph of ln function has:

(a) A vertical asymptote at $x = 0$.

(b) No horizontal asymptote.



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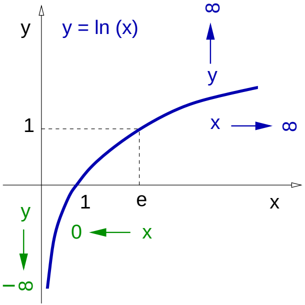

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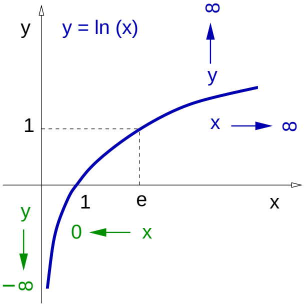

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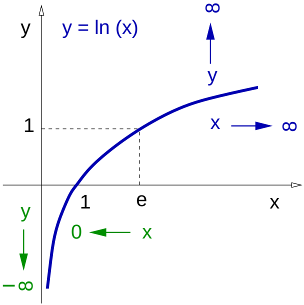

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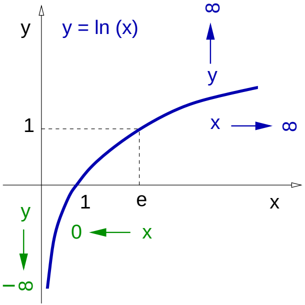

Proof: Recall $e = 2.718281... > 1$ and $\ln(e) = 1$.

(a): If $x = e^n$, then $\ln(e^n) = n \ln(e) = n$.

The graph of the natural logarithm

Remarks:

The graph of ln function has:

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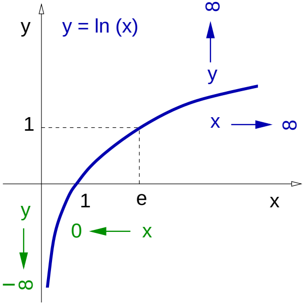

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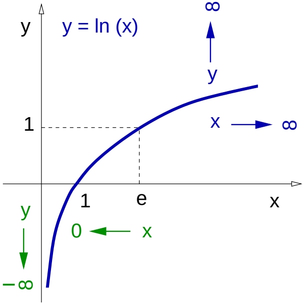

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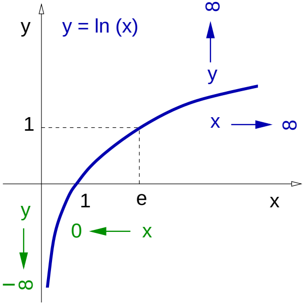

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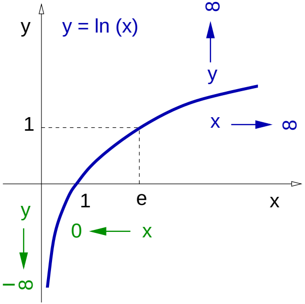

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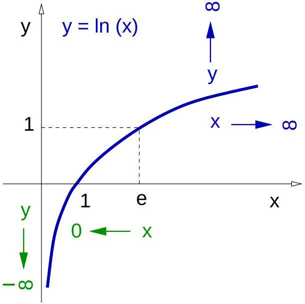

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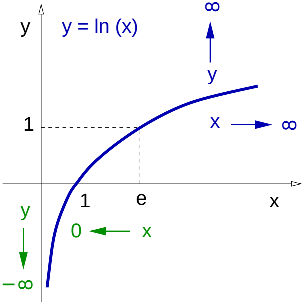

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Natural Logarithms (Sect. 7.2)

- Definition as an integral.
- The derivative and properties.
- The graph of the natural logarithm.
- **Integrals involving logarithms.**
- Logarithmic differentiation.

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Remark: It holds $\displaystyle\int \frac{dx}{x} = \ln(|x|) + c$ for $x \neq 0$ and $c \in \mathbb{R}$.

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$$\int \frac{dx}{x} = \begin{cases} \ln(-x) + c & \text{if } x < 0, \\ \ln(x) + c & \text{if } x > 0. \end{cases}$$

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Remark: It also holds $\displaystyle\int \frac{f'(x)}{f(x)}\, dx = \ln(|f(x)|) + c$, for $f(x) \neq 0$.

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We conclude that $y(t) = -3\,\ln\big(|2+\cos(t)|\big) + c.$ $\triangleleft$

Natural Logarithms (Sect. 7.2)

- Definition as an integral.
- The derivative and properties.
- The graph of the natural logarithm.
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- **Logarithmic differentiation.**

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The inverse function (Sect. 7.1)

- One-to-one functions.
- The inverse function
- The graph of the inverse function.
- Derivatives of the inverse function.

One-to-one functions

Remark:

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A function $f : D \to \mathbb{R}$ is called *one-to-one* (injective) iff for every $x_1,\, x_2 \in D$ holds

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3. $y = \sqrt{x}$, for $x \in [0, \infty)$.

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3. $y = \sqrt{x}$, for $x \in [0, \infty)$.
4. $y = \sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

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A function $f : D \to \mathbb{R}$ is called *one-to-one* (injective) iff for every $x_1, x_2 \in D$ holds

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Example

Invertible:

1. $y = x^3$, for $x \in \mathbb{R}$.
2. $y = x^2$, for $x \in [0, b]$.
3. $y = \sqrt{x}$, for $x \in [0, \infty)$.
4. $y = \sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Not Invertible:

1. $y = x^2$, for $x \in [-a, a]$.

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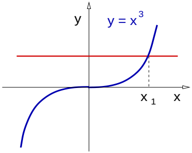

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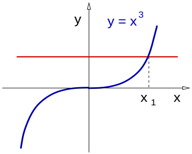

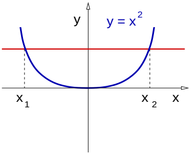

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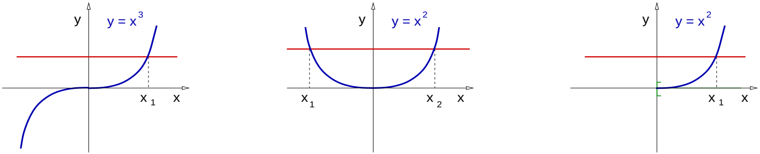

The inverse function (Sect. 7.1)

- One-to-one functions.
- **The inverse function**
- The graph of the inverse function.
- Derivatives of the inverse function.

The inverse function

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The *inverse* of a one-to-one function $f : D \to R$ is the function $f^{-1} : R \to D$ defined for all $x \in D$ and all $y \in R$ as follows

$$f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x).$$

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Find the inverse of $f(x) = 2x - 3$.

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Example

Find the inverse of $f(x) = 2x - 3$.

Solution: Denote $y = f(x)$, that is, $y = 2x - 3$.

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Find the inverse of $f(x) = 2x - 3$.

Solution: Denote $y = f(x)$, that is, $y = 2x - 3$. Find x in the expression above,

$$2x = y + 3$$

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Find the inverse of $f(x) = 2x - 3$.

Solution: Denote $y = f(x)$, that is, $y = 2x - 3$. Find x in the expression above,

$$2x = y + 3 \quad \Rightarrow \quad x = \frac{1}{2}\,y + \frac{3}{2}.$$

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$$2x = y + 3 \quad \Rightarrow \quad x = \frac{1}{2}\,y + \frac{3}{2}.$$

Then, the inverse function is $f^{-1}(y) = \frac{1}{2}\,y + \frac{3}{2}$. $\triangleleft$

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Remark:

- If f^{-1} is the inverse of f, then holds

$$(f^{-1} \circ f)(x) = x, \quad (f \circ f^{-1})(y) = y.$$

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Verify the relations above for $f(x) = 2x - 3$.

Solution: Recall: $f^{-1}(y) = (y+3)/2$.

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Example

Verify the relations above for $f(x) = 2x - 3$.

Solution: Recall: $f^{-1}(y) = (y + 3)/2$. Hence

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The inverse function (Sect. 7.1)

- One-to-one functions.
- The inverse function
- **The graph of the inverse function.**
- Derivatives of the inverse function.

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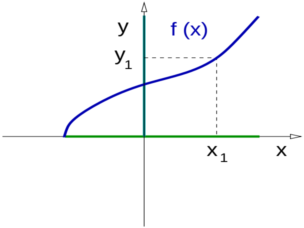

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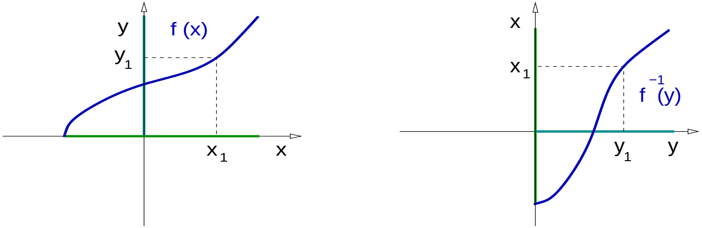

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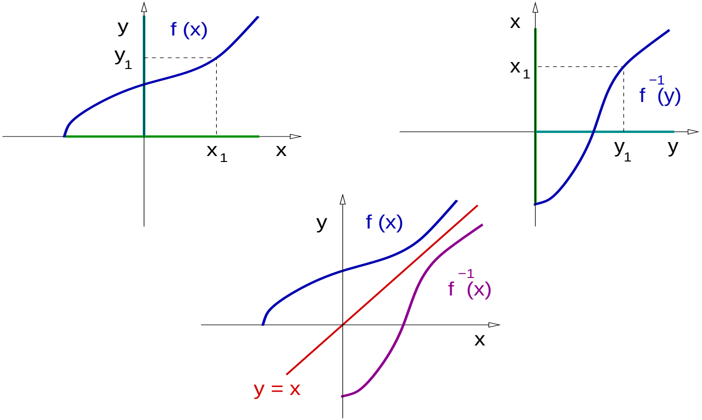

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Remark: The derivative values of a function and its inverse are deeply related.

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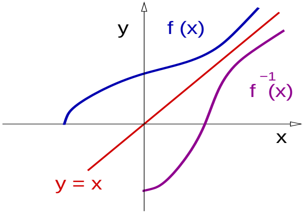

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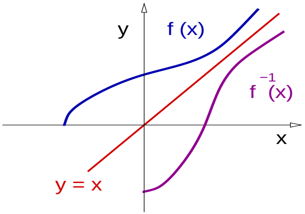

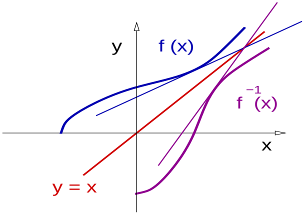

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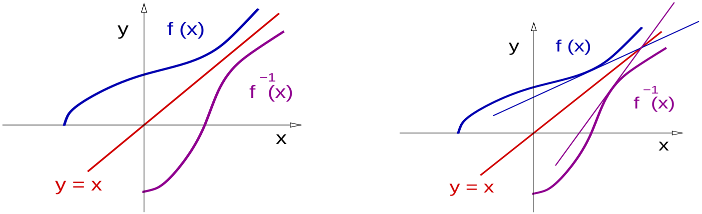

Theorem (Derivative for inverse functions)

If the invertible function $f : D \to R$ is differentiable and $f'(x) \neq 0$ for every $x \in D$, then the function $f^{-1} : R \to D$ is differentiable.

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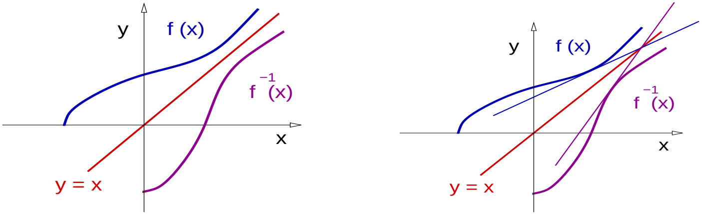

Theorem (Derivative for inverse functions)

If the invertible function $f : D \to R$ is differentiable and $f'(x) \neq 0$ for every $x \in D$, then the function $f^{-1} : R \to D$ is differentiable. Furthermore, for every $y \in R$ holds

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Since $f^{-1}(y) = \dfrac{1}{2}(y+3)$, then $\left(f^{-1}\right)'(y) = \dfrac{1}{2}$, constant.

Therefore, $\left(f^{-1}\right)' = \dfrac{1}{f'}$. ◁

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Example

Verify the Theorem above for $f(x) = x^3$.

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We first compute the inverse function. Denote $y = f(x)$, then

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Compute now the derivative of the inverse function,

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Derivatives of the inverse function.

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We conclude that $\left(f^{-1}\right)'(y) = \dfrac{1}{13}$. $\triangleleft$