Volumes as integrals of cross-sections (Sect. 6.1)

- The volume of simple regions in space
- Volumes integrating cross-sections:
  - The general case.
  - Regions of revolution.
  - Certain regions with holes.

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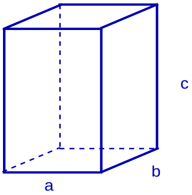

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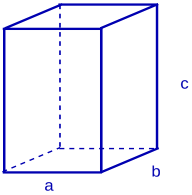

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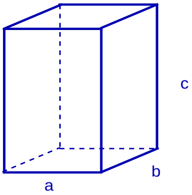

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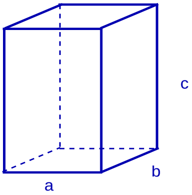

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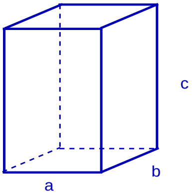

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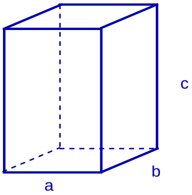

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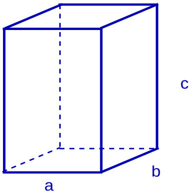

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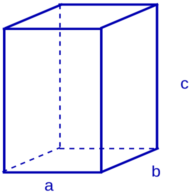

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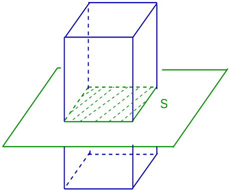

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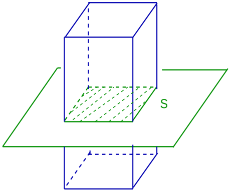

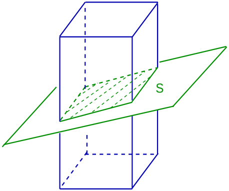

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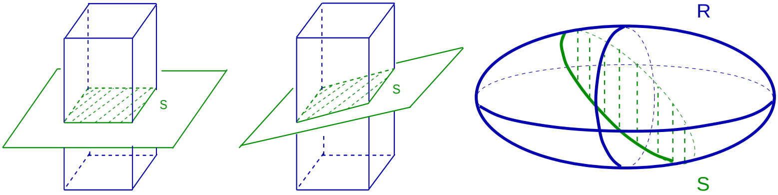

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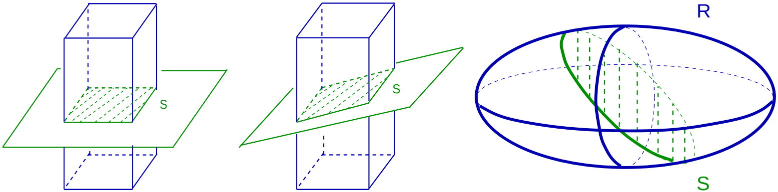

Remark: Like in the last case above, the area of a cross section is a function of the direction normal to the cross-section.

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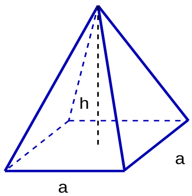

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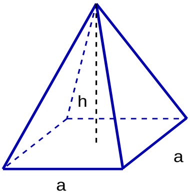

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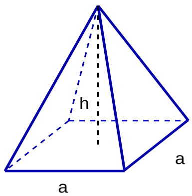

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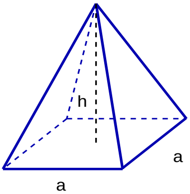

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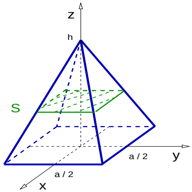

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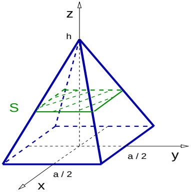

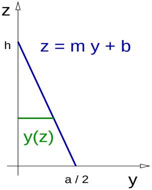

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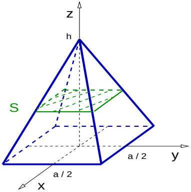

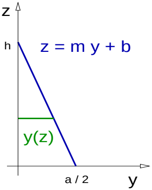

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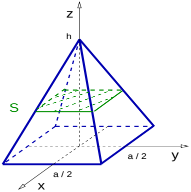

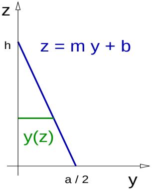

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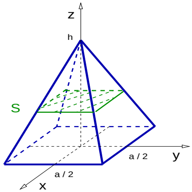

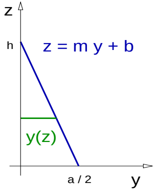

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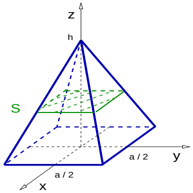

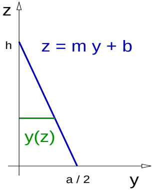

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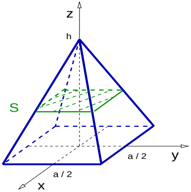

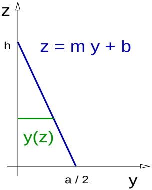

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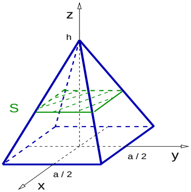

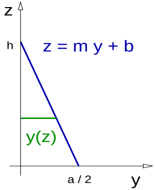

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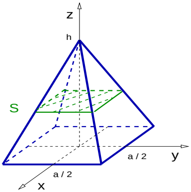

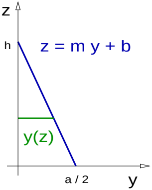

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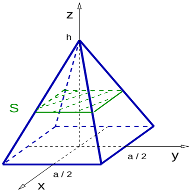

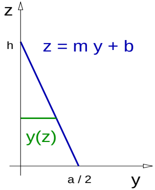

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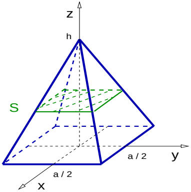

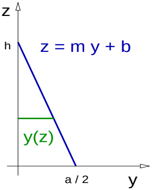

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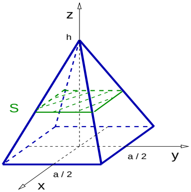

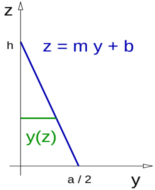

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$\lhd$

The volume of simple regions in space

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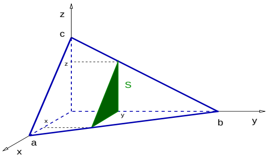

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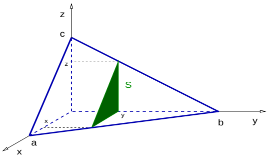

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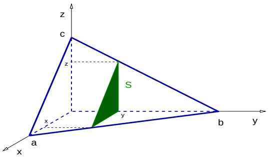

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The volume of simple regions in space

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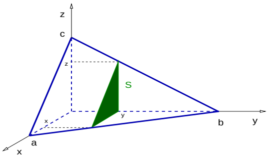

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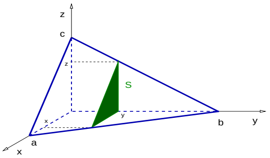

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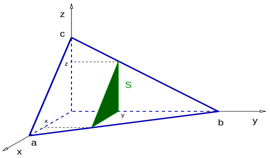

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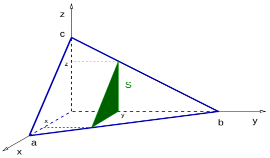

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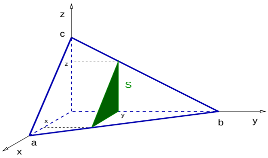

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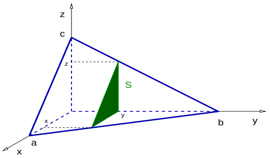

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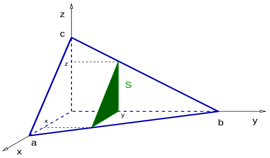

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◁

Volumes as integrals of cross-sections (Sect. 6.1)

- The volume of simple regions in space
- **Volumes integrating cross-sections:**
  - The general case.
  - **Regions of revolution.**
  - Certain regions with holes.

Regions of revolution

Definition

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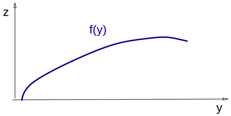

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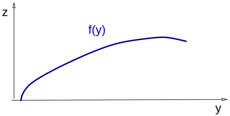

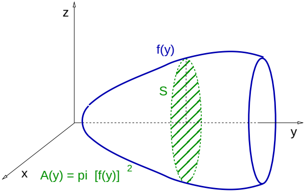

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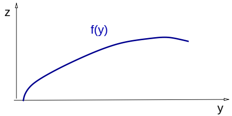

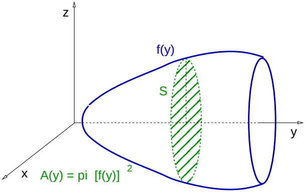

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- The cross-sections of region of revolution are disks: $A = \pi R^2$.

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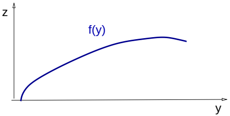

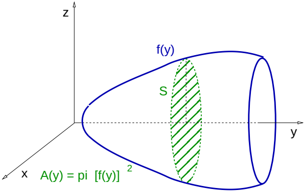

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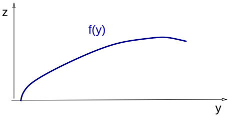

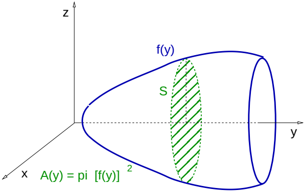

Remark:

- The cross-sections of region of revolution are disks: $A = \pi R^2$.
- In the example, $R(y) = f(y)$. Therefore, $A(y) = \pi \big[f(y) \big]^2$.

Regions of revolution

Theorem

The volume of a region of revolution defined by rotating the function values $z = f(y)$ *for* $y \in [a, b]$ *about the y-axis is*

$$V = \pi \int_a^b \left[f(y)\right]^2 dy.$$

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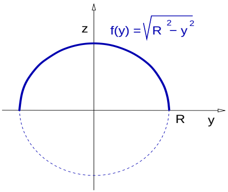

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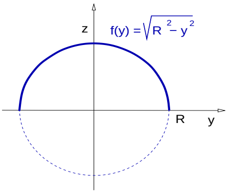

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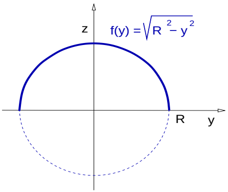

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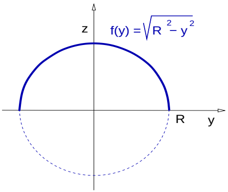

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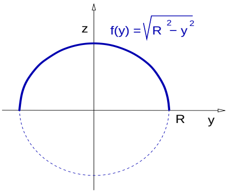

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$$V = \pi \Big[2R^3 - \frac{2}{3} R^3 \Big] \Rightarrow V = \frac{4}{3} \pi R^3. \ \triangleleft$$

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Remark: The axis of rotation could be any axis in space.

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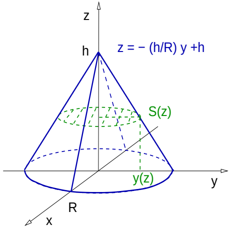

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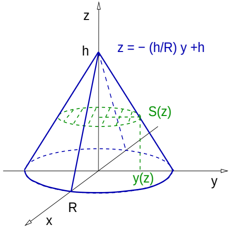

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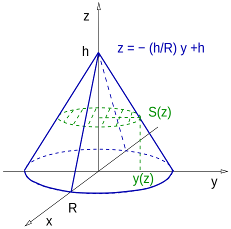

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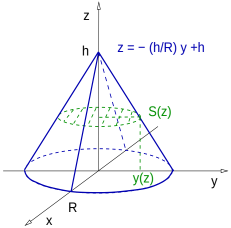

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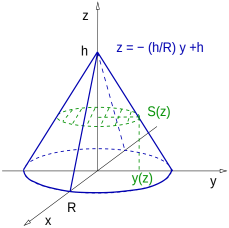

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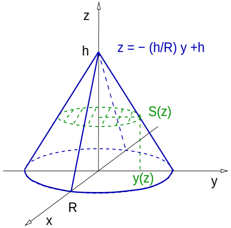

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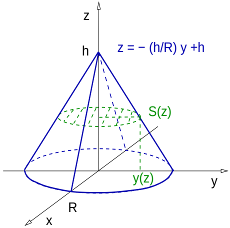

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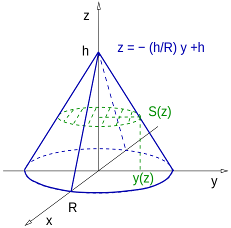

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We conclude that $V = \dfrac{1}{3}\pi R^2 h$. ◁

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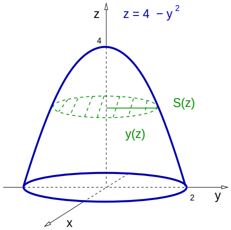

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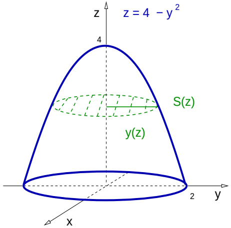

$$V = \pi \int_0^4 \left[y(z)\right]^2 dz.$$

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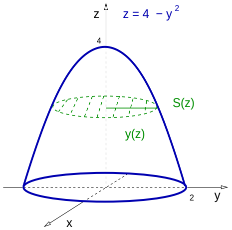

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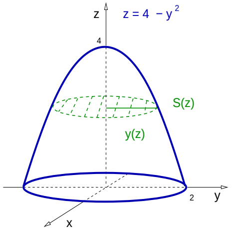

$$V = \pi \int_0^4 \big[y(z)\big]^2 \, dz.$$

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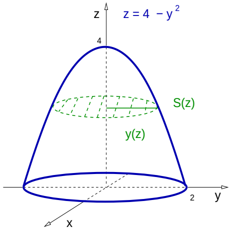

$$V = \pi \int_0^4 \big[y(z)\big]^2 \, dz.$$

Since $y^2 = -z + 4$, we get

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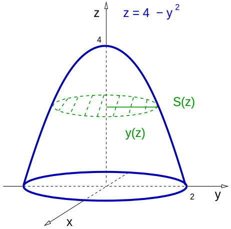

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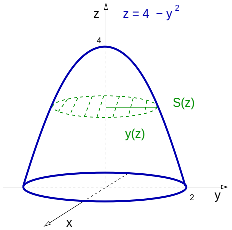
Find the volume of the region bounded by $z = -y^2 + 4$ for $y \in [0, 2]$ when it is rotated about the z axis.

Solution:



$$V = \pi \int_0^4 \big[y(z)\big]^2 \, dz.$$

Since $y^2 = -z + 4$, we get

$$V = \pi \int_0^4 (-z + 4)\, dz = \pi \Big[\frac{(-z^2)}{2} \Big|_0^4 + (4z)\Big|_0^4 \Big] = \pi[-8 + 16]$$

We conclude that $V = 8\pi$. $\lhd$

Volumes as integrals of cross-sections (Sect. 6.1)

- The volume of simple regions in space
- **Volumes integrating cross-sections:**
  - The general case.
  - Regions of revolution.
  - **Certain regions with holes.**

Certain regions with holes (washer method)

Definition

A *washer region* is a region of revolution with a hole, where the exterior and interior surfaces are obtained by rotating the function values $z = f_{ext}(y)$ and $z = f_{int}(y)$ along the y axis.

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Example

Sketch the washer region bounded by $z = -2y + 4$ and $z = -y^2 + 4$ for $y \in [0, 2]$, rotated about the z-axis.

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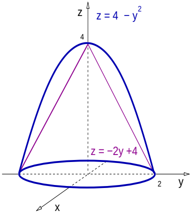

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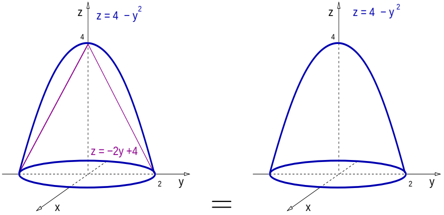

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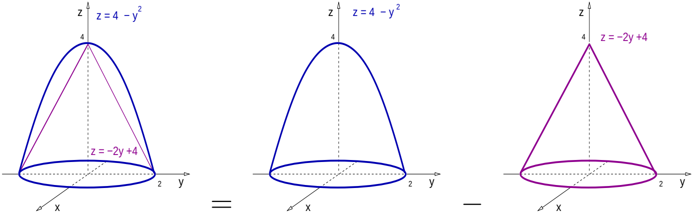

Certain regions with holes (washer method)

Theorem

The volume of a washer region about the z-axis with exterior and interior surfaces generated by $z = f_{ext}(y)$ *and* $z = f_{int}(y)$ *for* $y \in [a, b]$*, respectively, is given by*

$$V = V_{ext} - V_{int} \quad \Leftrightarrow \quad V = \pi \int_a^b \Big(\big[f_{ext}(y)\big]^2 - \big[f_{int}(y)\big]^2 \Big)\, dy.$$

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$$V = \Big(\frac{1}{2} - \frac{1}{3} \Big) 16\pi \quad \Rightarrow \quad V = \frac{8}{3}\pi. \qquad \lhd$$

The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- Curves with vertical asymptotes.
- The arc-length function.

The main length formula

Remark: The length of a straight segment can be obtained with Pythagoras Theorem.

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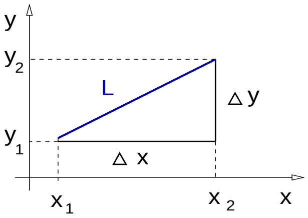
$$L = \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$



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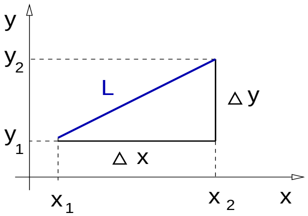

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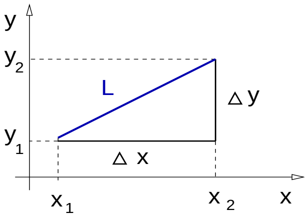

Remark: Calculus is needed to compute, and even define, the length of non-straight curves, called arc-length.

Definition

The *arc-length* of a curve in the plane given by a differentiable function $y = f(x)$, for $x \in [a, b]$, is

$$L = \int_a^b \sqrt{1 + \big[f'(x)\big]^2}\, dx.$$

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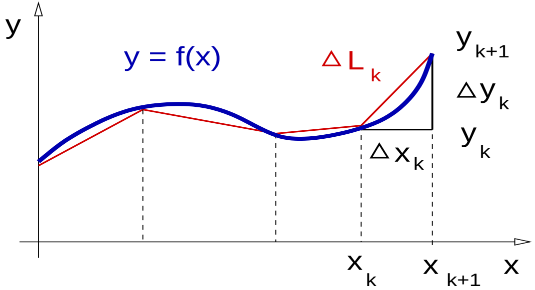

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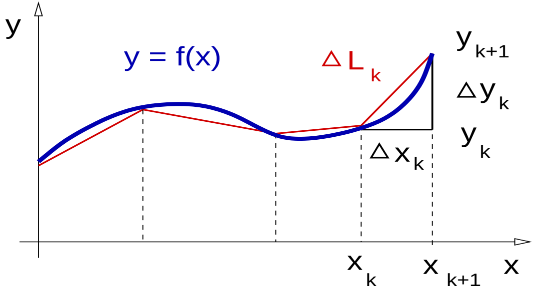

The length of the curve will be approximated by the red lines,

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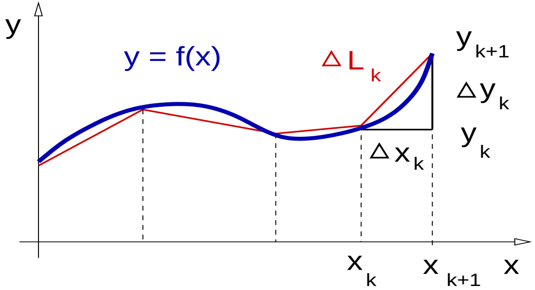

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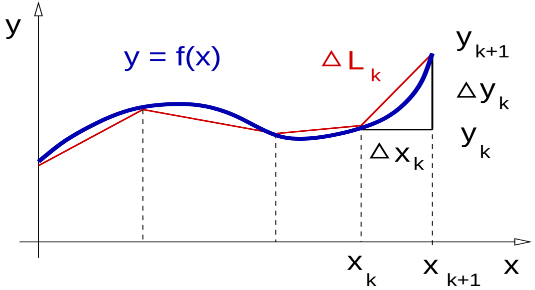

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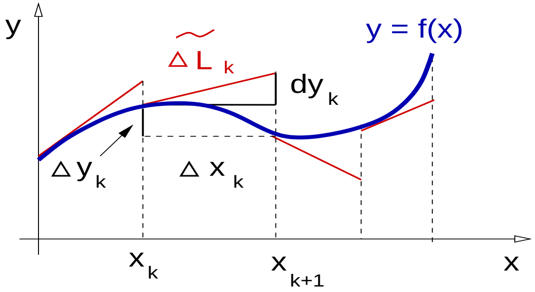
(b)



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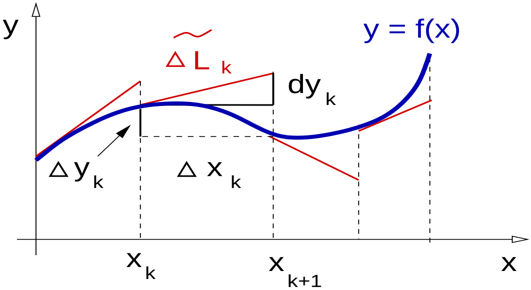

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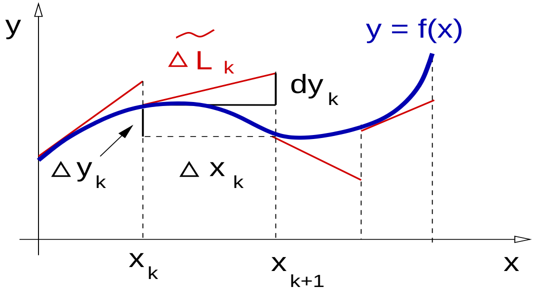

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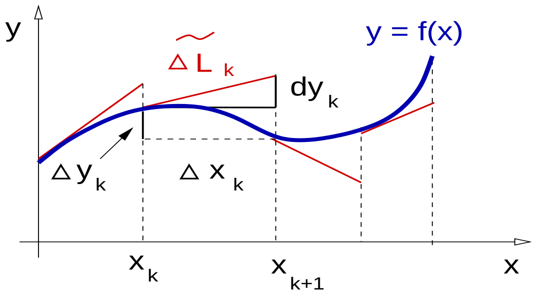

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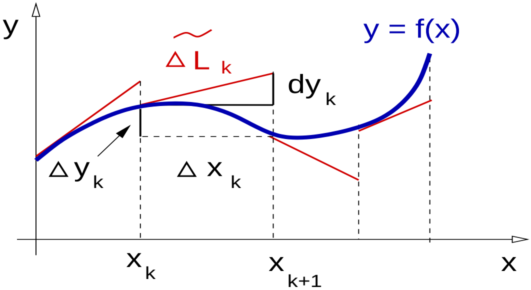

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We conclude that $L = \dfrac{8}{27}(10^{3/2} - 1)$. $\lhd$

The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- **Curves with vertical asymptotes.**
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Hence, $y'(x) \to \infty$ as $x \to 1^+$.

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Hence, $y'(x) \to \infty$ as $x \to 1^+$. Therefore, it is not clear how to compute

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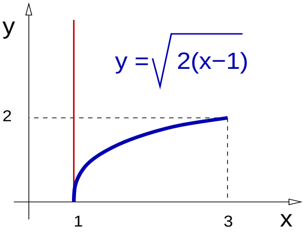

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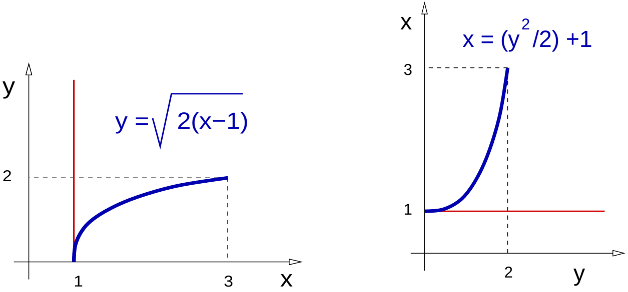

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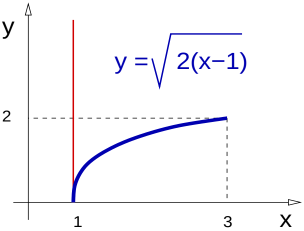

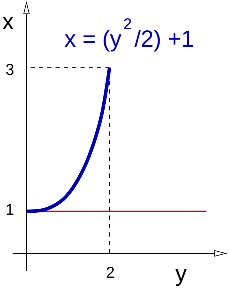

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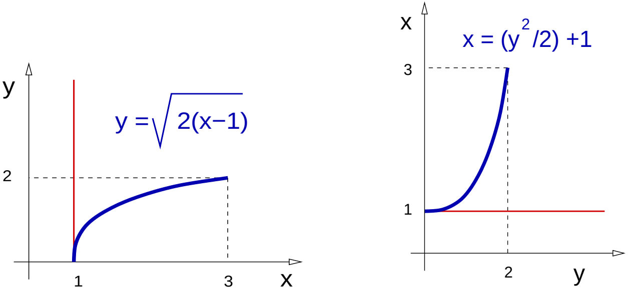

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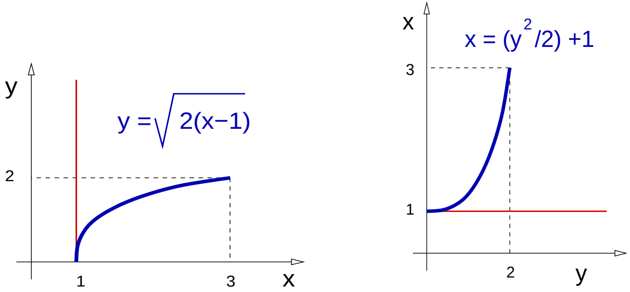

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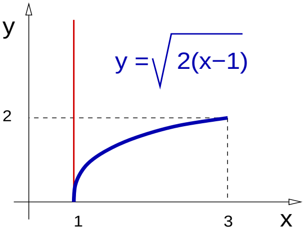

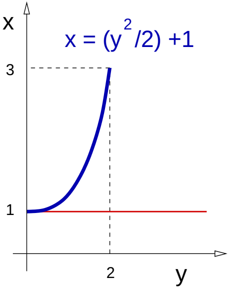

We now use $L = \int_0^2 \sqrt{1 + \left[x'(y)\right]^2}\, dy$. Since $x'(y) = y$,

$$L = \int_0^2 \sqrt{1 + y^2}\, dy = \left[\frac{y}{2}\sqrt{1 + y^2} + \frac{1}{2}\ln\left(y + \sqrt{1 + y^2}\right)\right]\Big|_0^2.$$

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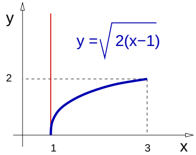

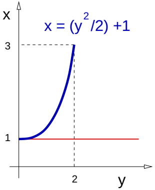

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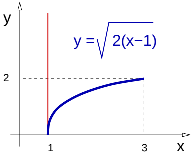

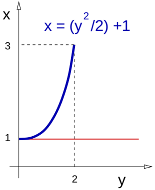

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We conclude that $L = \sqrt{5} + \frac{1}{2}\ln\left(2+\sqrt{5}\right)$. $\lhd$

The arc-length of curves in the plane (Sect. 6.3)

- The main arc-length formula.
- Curves with vertical asymptotes.
- **The arc-length function.**

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Remark: It is useful to introduce a function that measures a curve arc-length from a fix starting point to any other point in the curve.

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The *arc-length function* of a differentiable curve $y = f(x)$, for $x \in [a, b]$ is given by

$$L(x) = \int_a^x \sqrt{1 + \big[f'(\hat{x})\big]^2}\, d\hat{x}.$$

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We conclude that $L(x) = \dfrac{8}{27}\Big[\Big(1 + \dfrac{9}{4}\,x\Big)^{3/2} - 1\Big].$ $\triangleleft$

Work on solids and fluids (Sect. 6.5)

- Moving things around.
- Forces made by springs.
- Pumping liquids.

Moving Things around: Constant forces

Remarks:

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Moving Things around: Variable forces

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Moving Things around: Variable forces

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The *work* done on a particle moving on the x-axis by a non-constant force F along the x-axis for $x \in [a, b]$ is

$$W = \int_a^b F(x)\, dx.$$

Remarks:

- The formula above is obtained in the standard way: Introduce a partition in $[a, b]$ and compute the limit of partial sums

$$W_N = \sum_{k=0}^{N-1} F(x_k)\, \Delta x_k.$$

- The simplest variable force is the one produced by a spring. In 1660 Robert Hooke discovered that $F(x) = kx$, where k is called the spring constant,

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Work on solids and fluids (Sect. 6.5)

- Moving things around.
- **Forces made by springs.**
- Pumping liquids.

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Work on solids and fluids (Sect. 6.5)

- Moving things around.
- Forces made by springs.
- **Pumping liquids.**

Pumping liquids

Remark: Pumping liquids in or out an arbitrary shaped container is a typical problem with variable forces.

Theorem

Consider an arbitrary shaped container with horizontal cross section area $A(z)$, for $z \in [0, h]$, and let $g = 9.81\ m/s^2$.

(a) *If a liquid of density δ Kgr/m^3 is resting at the bottom of the container, then the work done to pump the liquid in the container, initially empty, up to a height $h_1 \leqslant h$ is*

$$W = \int_0^{h_1} \delta\, g\, A(z)\, z\, dz.$$

(b) *The work done to pump the liquid out from the top of a container, initially filled with liquid up to a height $h_1 \leqslant h$ is*

$$W = \int_0^{h_1} \delta\, g\, A(z)\, (h - z)\, dz.$$

Pumping liquids

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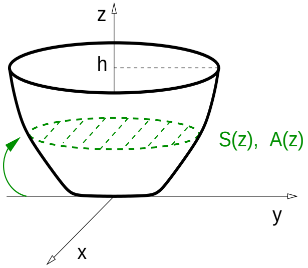

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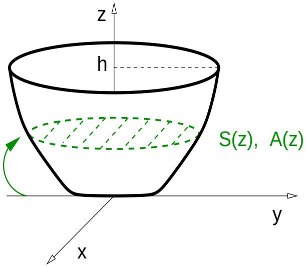

The amount of liquid that can be placed at cross-section $S(z)$ is

Pumping liquids

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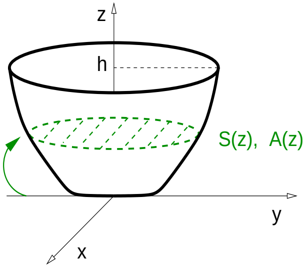

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$$L = \delta \, A(z) \, dz.$$

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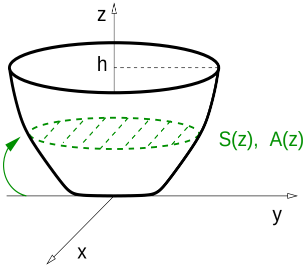

The amount of liquid that can be placed at cross-section $S(z)$ is

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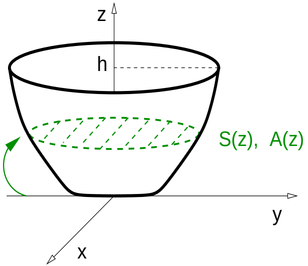

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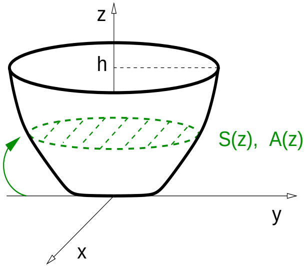

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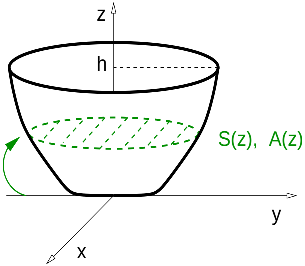

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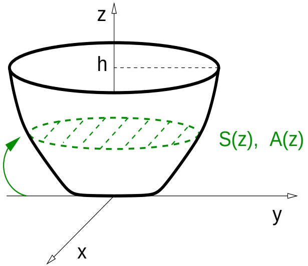

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The work to fill in the container up to h_1 is $W = \int_0^{h_1} \delta\, g\, A(z)\, z\, dz$.

Pumping liquids

Proof: (b) Show: $W = \int_0^{h_1} \delta\, g\, A(z)\,(h - z)\, dz.$

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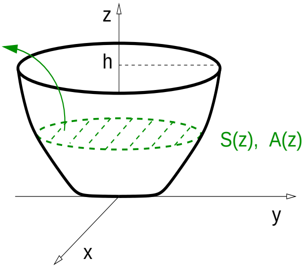

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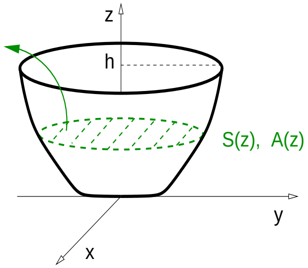

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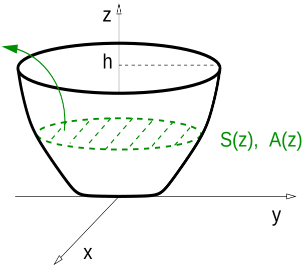

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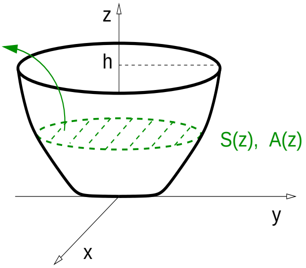

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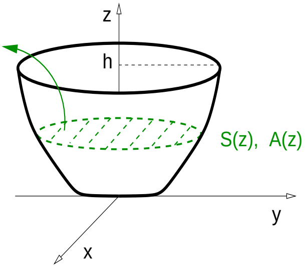

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The work to empty the container initially filled up to h_1 is

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Pumping liquids

Example

A tank has the shape of an inverted circular cone with height $h = 10\ m$ and base radius $R = 4\ m$. It is filled with water to a height $h_1 = 8\ m$. Recalling that the water density is $1\ gr/cm^3 = 1000\ Kgr/m^3$, find the work required to empty the tank pumping the water from the top.

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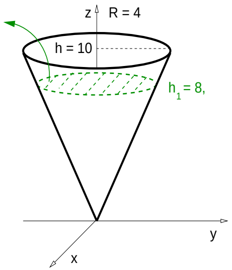

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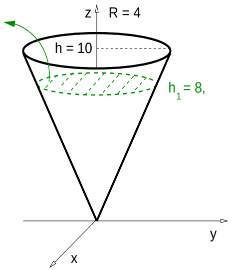

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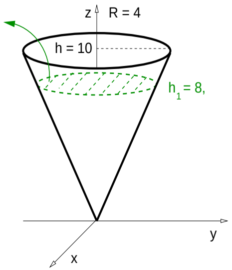

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Here $A(z) = \pi \big[R(z)\big]^2$

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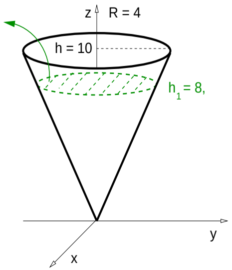

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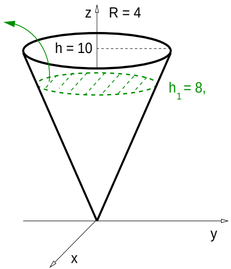

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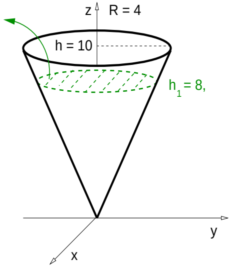

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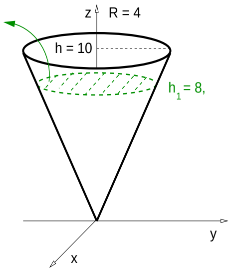

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$$W = \delta g \pi \frac{4}{25} \int_0^8 z^2 (10 - z)\, dz.$$

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$$W = \delta g \pi \frac{4}{25}\, 8^3\, \frac{4}{3}$$

Pumping liquids

Example

A tank has the shape of an inverted circular cone with height $h = 10\ m$ and base radius $R = 4\ m$. It is filled with water to a height $h_1 = 8\ m$. Recalling that the water density is $1\ gr/cm^3 = 1000\ Kgr/m^3$, find the work required to empty the tank pumping the water from the top.

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That is, $W = 3.4\, x\, 10^6\ J$. ◁