Mathematics 133
Supplementary Material Section 8.4

LEGO pieces for Integrals of Rational Functions

Function	Trick	Answer
$\int \frac{1}{a+bx}\,dx$	Substitution $u = a + bx$	$\frac{1}{b}\ln\lvert a+bx\rvert + C$
$\int \frac{1}{(a+bx)^n}\,dx,\ n>1$	Substitution $u = a + bx$	$-\frac{1}{(n-1)b(a+bx)^{n-1}} + C$
$\int \frac{x}{1+x^2}\,dx$	Substitution $u = 1 + x^2$	$\frac{1}{2}\ln(1+x^2) + C$
$\int \frac{x}{(1+x^2)^n}\,dx,\ n>1$	Substitution $u = 1 + x^2$	$\int \frac{x}{(1+x^2)^n}\,dx = \int \frac{1/2\,du}{u^n} =$ $= -\frac{1}{2(n-1)u^{n-1}} + C = -\frac{1}{2(n-1)(1+x^2)^{n-1}} + C$
$\int \frac{1}{1+x^2}\,dx$	Table Integral	$\tan^{-1}(x) + C$
$\int \frac{1}{(1+x^2)^n}\,dx\; n>1$	Substitution $x = tanz$	$\int \frac{1}{(1+x^2)^n}\,dx = \int \frac{\sec^2(z)\,dz}{sec^{2n}(z)} =$ $= \int \frac{dz}{sec^{2(n-1)}(z)} = \int \cos^{2(n-1)}(z)\,dz$

To integrate any rational function we do two steps.

- Using partial fraction method we express any rational function as a sum of simple fractions with linear or quadratic denominators and a pure polynomial function.
- Integrate the obtained sum term by term using the "LEGO pieces" table above. Namely, a term with some power of linear function in denominator can be integrated as shown in the first two rows of the table. Any quadratic denominator can be transformed into denominator like a power of $1 + x^2$ using completion of square method. Then the corresponding integration method is shown in the last four rows of the table.

Example

$$\int \frac{x^2+1}{x^2+2x+2}\,dx = \int \left(1 - \frac{2x+1}{(x+1)^2+1}\right)\,dx =$$

$$= \int dx - \int \frac{2(x+1)}{(x+1)^2+1}\,dx + \int \frac{1}{(x+1)^2+1}\,dx = x - \ln((x+1)^2+1) + \tan^{-1}(x+1) + C$$