



Chapter

8

TECHNIQUES OF INTEGRATION

OVERVIEW The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x) dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C :

$$\int f(x) dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1

Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x) dx = \int f(u) du$$

where $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I . Success in integration often hinges on the ability to spot what part of the integrand should be called u in order that one will also have du , so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$	13. $\int \cot u \, du = \ln \sin u + C$ $= -\ln \csc u + C$
2. $\int k \, du = ku + C$ (any number k)	14. $\int e^u \, du = e^u + C$
3. $\int (du + dv) = \int du + \int dv$	15. $\int a^u \, du = \frac{a^u}{\ln a} + C$ ($a > 0, a \neq 1$)
4. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C$ ($n \neq -1$)	16. $\int \sinh u \, du = \cosh u + C$
5. $\int \frac{du}{u} = \ln u + C$	17. $\int \cosh u \, du = \sinh u + C$
6. $\int \sin u \, du = -\cos u + C$	18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \left(\frac{u}{a} \right) + C$
7. $\int \cos u \, du = \sin u + C$	19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \left(\frac{u}{a} \right) + C$
8. $\int \sec^2 u \, du = \tan u + C$	20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left \frac{u}{a} \right + C$
9. $\int \csc^2 u \, du = -\cot u + C$	21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1} \left(\frac{u}{a} \right) + C$ ($a > 0$)
10. $\int \sec u \tan u \, du = \sec u + C$	22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1} \left(\frac{u}{a} \right) + C$ ($u > a > 0$)
11. $\int \csc u \cot u \, du = -\csc u + C$	
12. $\int \tan u \, du = -\ln \cos u + C$ $= \ln \sec u + C$	

We often have to rewrite an integral to match it to a standard formula.

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} \, dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx = \int \frac{du}{\sqrt{u}}$$

$$u = x^2 - 9x + 1, \\ du = (2x - 9) dx.$$

$$= \int u^{-1/2} du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2)+1} + C$$

Table 8.1 Formula 4,
with $n = -1/2$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

EXAMPLE 2 Completing the Square

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}$$

Solution We complete the square to simplify the denominator:

$$8x - x^2 = -(x^2 - 8x) = -(x^2 - 8x + 16 - 16)$$

$$= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2.$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$$a = 4, u = (x - 4), \\ du = dx$$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

Table 8.1, Formula 18

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Evaluate

$$\int (\sec x + \tan x)^2 dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2 \sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \quad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned} \int (\sec x + \tan x)^2 dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1) dx \\ &= 2 \int \sec^2 x dx + 2 \int \sec x \tan x dx - \int 1 dx \\ &= 2 \tan x + 2 \sec x - x + C. \end{aligned}$$

EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x} dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \quad \text{or} \quad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned} \int_0^{\pi/4} \sqrt{1 + \cos 4x} dx &= \int_0^{\pi/4} \sqrt{2} \sqrt{\cos^2 2x} dx \\ &= \sqrt{2} \int_0^{\pi/4} |\cos 2x| dx \\ &= \sqrt{2} \int_0^{\pi/4} \cos 2x dx \\ &= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} \\ &= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}. \end{aligned}$$

$$\sqrt{u^2} = |u|$$

On $[0, \pi/4]$, $\cos 2x \geq 0$,
so $|\cos 2x| = \cos 2x$.

Table 8.1, Formula 7, with
 $u = 2x$ and $du = 2 dx$

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2} dx.$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2} dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C. \quad \blacksquare$$

$$\begin{array}{r} x - 3 \\ 3x + 2 \overline{) 3x^2 - 7x} \\ \underline{3x^2 + 2x} \\ -9x \\ \underline{-9x - 6} \\ + 6 \end{array}$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = 3 \int \frac{x dx}{\sqrt{1 - x^2}} + 2 \int \frac{dx}{\sqrt{1 - x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \quad du = -2x dx, \quad \text{and} \quad x dx = -\frac{1}{2} du.$$

$$\begin{aligned} 3 \int \frac{x dx}{\sqrt{1 - x^2}} &= 3 \int \frac{(-1/2) du}{\sqrt{u}} = -\frac{3}{2} \int u^{-1/2} du \\ &= -\frac{3}{2} \cdot \frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1 - x^2} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2 \int \frac{dx}{\sqrt{1 - x^2}} = 2 \sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x + 2}{\sqrt{1 - x^2}} dx = -3\sqrt{1 - x^2} + 2 \sin^{-1} x + C. \quad \blacksquare$$

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

EXAMPLE 7 Integral of $y = \sec x$ —Multiplying by a Form of 1

Evaluate

$$\int \sec x dx.$$

Solution

$$\begin{aligned} \int \sec x dx &= \int (\sec x)(1) dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} dx \\ &= \int \frac{du}{u} \\ &= \ln |u| + C = \ln |\sec x + \tan x| + C. \end{aligned}$$

$$\begin{aligned} u &= \sec x + \tan x, \\ du &= (\sec^2 x + \sec x \tan x) dx \end{aligned}$$

HISTORICAL BIOGRAPHY

George David Birkhoff
(1884–1944)

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

TABLE 8.2 The secant and cosecant integrals

1. $\int \sec u \, du = \ln |\sec u + \tan u| + C$
2. $\int \csc u \, du = -\ln |\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE

EXAMPLE

Making a simplifying substitution

$$\frac{2x - 9}{\sqrt{x^2 - 9x + 1}} dx = \frac{du}{\sqrt{u}}$$

Completing the square

$$\sqrt{8x - x^2} = \sqrt{16 - (x - 4)^2}$$

Using a trigonometric identity

$$\begin{aligned} (\sec x + \tan x)^2 &= \sec^2 x + 2 \sec x \tan x + \tan^2 x \\ &= \sec^2 x + 2 \sec x \tan x \\ &\quad + (\sec^2 x - 1) \\ &= 2 \sec^2 x + 2 \sec x \tan x - 1 \end{aligned}$$

Eliminating a square root

$$\sqrt{1 + \cos 4x} = \sqrt{2 \cos^2 2x} = \sqrt{2} |\cos 2x|$$

Reducing an improper fraction

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}$$

Separating a fraction

$$\frac{3x + 2}{\sqrt{1 - x^2}} = \frac{3x}{\sqrt{1 - x^2}} + \frac{2}{\sqrt{1 - x^2}}$$

Multiplying by a form of 1

$$\begin{aligned} \sec x &= \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \\ &= \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \end{aligned}$$

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

$$1. \int \frac{16x \, dx}{\sqrt{8x^2 + 1}}$$

$$2. \int \frac{3 \cos x \, dx}{\sqrt{1 + 3 \sin x}}$$

$$3. \int 3\sqrt{\sin v} \cos v \, dv$$

$$4. \int \cot^3 y \csc^2 y \, dy$$

$$5. \int_0^1 \frac{16x \, dx}{8x^2 + 2}$$

$$6. \int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z} \, dz$$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x} + 1)}$ 8. $\int \frac{dx}{x - \sqrt{x}}$
9. $\int \cot(3 - 7x) dx$ 10. $\int \csc(\pi x - 1) dx$
11. $\int e^\theta \csc(e^\theta + 1) d\theta$ 12. $\int \frac{\cot(3 + \ln x)}{x} dx$
13. $\int \sec \frac{t}{3} dt$ 14. $\int x \sec(x^2 - 5) dx$
15. $\int \csc(s - \pi) ds$ 16. $\int \frac{1}{\theta^2} \csc \frac{1}{\theta} d\theta$
17. $\int_0^{\sqrt{\ln 2}} 2x e^{x^2} dx$ 18. $\int_{\pi/2}^{\pi} (\sin y) e^{\cos y} dy$
19. $\int e^{\tan v} \sec^2 v dv$ 20. $\int \frac{e^{\sqrt{t}} dt}{\sqrt{t}}$
21. $\int 3^{x+1} dx$ 22. $\int \frac{2^{\ln x}}{x} dx$
23. $\int \frac{2^{\sqrt{w}} dw}{2\sqrt{w}}$ 24. $\int 10^{2\theta} d\theta$
25. $\int \frac{9 du}{1 + 9u^2}$ 26. $\int \frac{4 dx}{1 + (2x + 1)^2}$
27. $\int_0^{1/6} \frac{dx}{\sqrt{1 - 9x^2}}$ 28. $\int_0^1 \frac{dt}{\sqrt{4 - t^2}}$
29. $\int \frac{2s ds}{\sqrt{1 - s^4}}$ 30. $\int \frac{2 dx}{x\sqrt{1 - 4 \ln^2 x}}$
31. $\int \frac{6 dx}{x\sqrt{25x^2 - 1}}$ 32. $\int \frac{dr}{r\sqrt{r^2 - 9}}$
33. $\int \frac{dx}{e^x + e^{-x}}$ 34. $\int \frac{dy}{\sqrt{e^{2y} - 1}}$
35. $\int_1^{e^{\pi/3}} \frac{dx}{x \cos(\ln x)}$ 36. $\int \frac{\ln x dx}{x + 4x \ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\int_1^2 \frac{8 dx}{x^2 - 2x + 2}$ 38. $\int_2^4 \frac{2 dx}{x^2 - 6x + 10}$
39. $\int \frac{dt}{\sqrt{-t^2 + 4t - 3}}$ 40. $\int \frac{d\theta}{\sqrt{2\theta - \theta^2}}$
41. $\int \frac{dx}{(x + 1)\sqrt{x^2 + 2x}}$ 42. $\int \frac{dx}{(x - 2)\sqrt{x^2 - 4x + 3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\int (\sec x + \cot x)^2 dx$ 44. $\int (\csc x - \tan x)^2 dx$
45. $\int \csc x \sin 3x dx$
46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x) dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\int \frac{x}{x + 1} dx$ 48. $\int \frac{x^2}{x^2 + 1} dx$
49. $\int_{\sqrt{2}}^3 \frac{2x^3}{x^2 - 1} dx$ 50. $\int_{-1}^3 \frac{4x^2 - 7}{2x + 3} dx$
51. $\int \frac{4t^3 - t^2 + 16t}{t^2 + 4} dt$ 52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5} d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\int \frac{1 - x}{\sqrt{1 - x^2}} dx$ 54. $\int \frac{x + 2\sqrt{x - 1}}{2x\sqrt{x - 1}} dx$
55. $\int_0^{\pi/4} \frac{1 + \sin x}{\cos^2 x} dx$ 56. $\int_0^{1/2} \frac{2 - 8x}{1 + 4x^2} dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\int \frac{1}{1 + \sin x} dx$ 58. $\int \frac{1}{1 + \cos x} dx$
59. $\int \frac{1}{\sec \theta + \tan \theta} d\theta$ 60. $\int \frac{1}{\csc \theta + \cot \theta} d\theta$
61. $\int \frac{1}{1 - \sec x} dx$ 62. $\int \frac{1}{1 - \csc x} dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\int_0^{2\pi} \sqrt{\frac{1 - \cos x}{2}} dx$ 64. $\int_0^{\pi} \sqrt{1 - \cos 2x} dx$

$$65. \int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t} dt$$

$$66. \int_{-\pi}^0 \sqrt{1 + \cos t} dt$$

$$67. \int_{-\pi}^0 \sqrt{1 - \cos^2 \theta} d\theta$$

$$68. \int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta} d\theta$$

$$69. \int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y} dy$$

$$70. \int_{-\pi/4}^0 \sqrt{\sec^2 y - 1} dy$$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

$$71. \int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2 dx$$

$$72. \int_0^{\pi/4} (\sec x + 4 \cos x)^2 dx$$

$$73. \int \cos \theta \csc(\sin \theta) d\theta$$

$$74. \int \left(1 + \frac{1}{x}\right) \cot(x + \ln x) dx$$

$$75. \int (\csc x - \sec x)(\sin x + \cos x) dx$$

$$76. \int 3 \sinh\left(\frac{x}{2} + \ln 5\right) dx$$

$$77. \int \frac{6 dy}{\sqrt{y(1+y)}}$$

$$78. \int \frac{dx}{x\sqrt{4x^2 - 1}}$$

$$79. \int \frac{7 dx}{(x-1)\sqrt{x^2 - 2x - 48}}$$

$$80. \int \frac{dx}{(2x+1)\sqrt{4x^2 + 4x}}$$

$$81. \int \sec^2 t \tan(\tan t) dt$$

$$82. \int \frac{dx}{x\sqrt{3+x^2}}$$

Trigonometric Powers

83. a. Evaluate $\int \cos^3 \theta d\theta$. (Hint: $\cos^2 \theta = 1 - \sin^2 \theta$.)
 b. Evaluate $\int \cos^5 \theta d\theta$.
 c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9 \theta d\theta$.
84. a. Evaluate $\int \sin^3 \theta d\theta$. (Hint: $\sin^2 \theta = 1 - \cos^2 \theta$.)
 b. Evaluate $\int \sin^5 \theta d\theta$.
 c. Evaluate $\int \sin^7 \theta d\theta$.
 d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13} \theta d\theta$.
85. a. Express $\int \tan^3 \theta d\theta$ in terms of $\int \tan \theta d\theta$. Then evaluate $\int \tan^3 \theta d\theta$. (Hint: $\tan^2 \theta = \sec^2 \theta - 1$.)
 b. Express $\int \tan^5 \theta d\theta$ in terms of $\int \tan^3 \theta d\theta$.
 c. Express $\int \tan^7 \theta d\theta$ in terms of $\int \tan^5 \theta d\theta$.
 d. Express $\int \tan^{2k+1} \theta d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1} \theta d\theta$.
86. a. Express $\int \cot^3 \theta d\theta$ in terms of $\int \cot \theta d\theta$. Then evaluate $\int \cot^3 \theta d\theta$. (Hint: $\cot^2 \theta = \csc^2 \theta - 1$.)

- b. Express $\int \cot^5 \theta d\theta$ in terms of $\int \cot^3 \theta d\theta$.
 c. Express $\int \cot^7 \theta d\theta$ in terms of $\int \cot^5 \theta d\theta$.
 d. Express $\int \cot^{2k+1} \theta d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1} \theta d\theta$.

Theory and Examples

87. **Area** Find the area of the region bounded above by $y = 2 \cos x$ and below by $y = \sec x$, $-\pi/4 \leq x \leq \pi/4$.
88. **Area** Find the area of the “triangular” region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \leq x \leq \pi/2$, and on the left by the line $x = \pi/6$.
89. **Volume** Find the volume of the solid generated by revolving the region in Exercise 87 about the x -axis.
90. **Volume** Find the volume of the solid generated by revolving the region in Exercise 88 about the x -axis.
91. **Arc length** Find the length of the curve $y = \ln(\cos x)$, $0 \leq x \leq \pi/3$.
92. **Arc length** Find the length of the curve $y = \ln(\sec x)$, $0 \leq x \leq \pi/4$.
93. **Centroid** Find the centroid of the region bounded by the x -axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.
94. **Centroid** Find the centroid of the region that is bounded by the x -axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.
95. **The integral of $\csc x$** Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x dx = -\ln |\csc x + \cot x| + C.$$

96. **Using different substitutions** Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3} dx$$

can be evaluated with any of the following substitutions.

- a. $u = 1/(x + 1)$
 b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1
 c. $u = \tan^{-1} x$
 d. $u = \tan^{-1} \sqrt{x}$ e. $u = \tan^{-1}((x - 1)/2)$
 f. $u = \cos^{-1} x$ g. $u = \cosh^{-1} x$

What is the value of the integral? (Source: “Problems and Solutions,” *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)