Series Tests — Complete Summary

### Standard Series

1. **Geometric Series**
   \[ \sum_{n=0}^{\infty} Ar^n = A + Ar + Ar^2 + \cdots = \begin{cases} 
   \frac{A}{1-r} & \text{if } |r| < 1 \\
   \text{diverges} & \text{if } |r| \geq 1 
\end{cases} \]

2. **p-Series**
   \[ \sum_{n=1}^{\infty} \frac{1}{n^p} \] converges if and only if \( p > 1 \) (e.g. \( \sum \frac{1}{n^2} \) converges, \( \sum \frac{1}{\sqrt{n}} \) diverges).

3. **Constant Series**
   \[ \sum_{n=0}^{\infty} c = c + c + c + \cdots \] diverges (unless \( c = 0 \)).

4. **Exponential Series**
   \[ \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x \] (converges for any \( x \) by the ratio test).

### Our Tests

1. **Integral Test:** If \( f(x) \) is a continuous, non-negative, decreasing function, then
   \[ \sum_{n=1}^{\infty} f(n) \] converges \( \iff \int_1^{\infty} f(x) \, dx \) is finite.

2. **Comparison Test:** If \( 0 \leq a_n \leq b_n \) for all large \( n \), then
   \[ \sum b_n \] converges \( \implies \sum a_n \) converges
   \[ \sum a_n \] diverges \( \implies \sum b_n \) diverges

3. **Limit Comparison Test:** If \( a_n, b_n \geq 0 \) and
   \[ \lim_{n \to \infty} \frac{a_n}{b_n} = L \] with \( L \neq 0 \) or \( \infty \)
   then \( \sum a_n \) and \( \sum b_n \) either both converge or both diverge.
   
   *This makes precise the intuition that \( a_n \approx L b_n \) for large \( n \). To apply it, take \( \sum b_n \) to be one of the “Standard Series” or one that can be handled with the integral test.*

4. **Ratio Test:** If \( a_n \geq 0 \) and \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = r \) then
   \[ \begin{cases} 
   \text{if } r < 1 & \text{then } \sum a_n \text{ converges absolutely} \\
   \text{if } r > 1 & \text{then } \sum a_n \text{ diverges} \\
   \text{if } r = 1 & \text{can’t tell} 
\end{cases} \]
   
   *This is useful for series involving exponentials (like \( 2^n \)) and factorials (like \( n! \)).*

5. **Alternating Series Test:** If the \( a_n \) are non-negative (\( a_n \geq 0 \)), decreasing (\( a_1 \geq a_2 \geq a_3 \geq \cdots \)), and \( \lim_{n \to \infty} a_n = 0 \), then \( \sum (-1)^n a_n \) converges.
Testing for Convergence

Check the convergence of a series $\sum a_n$ by the following steps.

1. Check that $\lim_{n \to \infty} |a_n| = 0$. If not the series diverges.

2. Check $\sum |a_n|$ by any test. If this converges $\sum a_n$ is absolutely convergent.

3. If the series is alternating (i.e. of the form $\sum (-1)^n|a_n|$) and the $|a_n|$ are decreasing (for example, if derivative $< 0$ for large $x$) then the series is conditionally convergent by the A.S.T.

Note: If you apply the the ratio or root test to $\sum |a_n|$ and get a limiting ratio $r > 1$, the series diverges and Step (3) is not needed.

For (2) ask yourself:
- Can I do the corresponding integral? If so, use the integral test.
- Can I simplify by dropping 'lower order terms'? If so, justify this simplification by the L.C.T.
- Try the ratio test — especially if the terms involve factorials.
- Can I find an inequality comparing $\sum a_n$ to a standard series? If so use the C.T.

**Power Series** $\sum a_n(x-a)^n$

Always apply the Ratio Test to the series $\sum |a_n(x-a)^n|$. The condition $\lim_{n \to \infty} \left| \frac{a_{n+1}(x-a)^{n+1}}{a_n(x-a)^n} \right| < 1$ gives the inequality $|x-a| < R$ for some $R$ (possibly $\infty$). The power series converges absolutely for each $x$ inside the interval $(a-R, a+R)$ and diverges for each $x$ outside the interval.

(The values $x = a \pm R$ on the boundary of this interval must be checked separately, but you won’t be asked to do this.)

**Taylor Series**

**Theorem** If $f(x)$ has $n+1$ derivatives on an interval $[a, x]$ then

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2} f''(a)(x-a)^2 + \ldots + \frac{1}{n!} f^{(n)}(a)(x-a)^n + R_n$$

where the remainder satisfies

$$|R_n| \leq M \frac{|x-a|^{n+1}}{(n+1)!}$$

where $M = \max.$ of $|f^{(n+1)}(t)|$ for $t$ between $a$ and $x$. 
Important Taylor Series

These series converge for any $x$:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

and these converge on the interval $|x| < 1$:

$$\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

$$\ln(1 - x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \cdots = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n.$$