Supplemental Material for Section 10.2: Convergence of an Infinite Series

The notion of a convergent infinite series is central to the rest of the material in Chapter 10. The idea is given a sequence of numbers $a_0, a_1, a_2, a_3 \ldots$ investigate the sum of these numbers; that is, $a_0 + a_1 + a_2 + a_3 + \ldots$. What should be clear is that if such a sum exists , then it can be approximated by $a_0 + a_1 + \cdots + a_k = \sum_{n=0}^{k} a_n$. For each positive integer k. So as k increases, the approximation gets better and better. The formal definition makes all of this precise using the notion of limit of a sequence.

Definition. *Let $\{a_n\}$ be a sequence (of terms). Then $\sum a_n$ converges means that the sequence $\left\{ \sum_{n=0}^{k} a_n \right\}$ has a limit that is a number. Otherwise we say that $\sum a_n$ diverges.*

The sequence $\{s_k\}$ defined by $s_k = \sum_{n=0}^{k} a_n$ is called the sequence of partial sums of the infinite series $\sum a_n$. So said another way $\sum a_n$converges means the sequence $\{s_k\}$ has a numerical limit. If $\sum a_n$ converges, then we let $\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} \sum_{n=0}^{k} a_n$.

To better understand the meaning of "$\sum a_n$ converges", let $\{a_n\}$ be a sequence of terms. Define a function f on the infinite interval $[0, \infty)$ by $f(x) = a_n$ for x in the interval $[n, n+1)$ for each positive integer n. The graph of f is given in Figure 1.

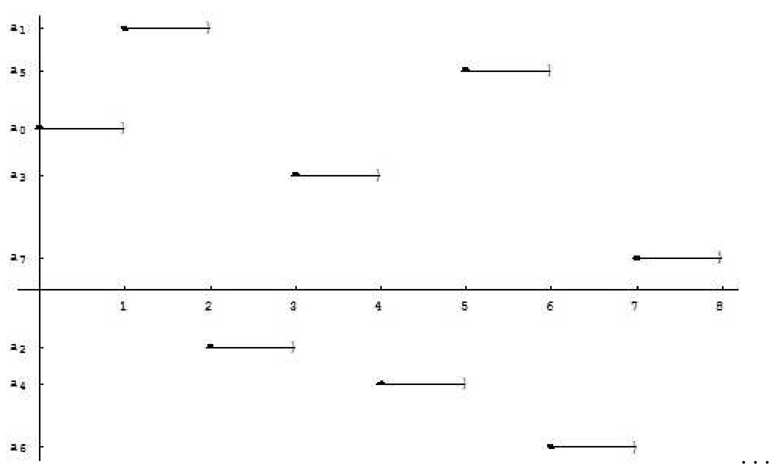

Figure 1: Graph of $y = f(x)$

For any positive integer k, $s_k = \sum_{n=0}^{k} a_n = \int_0^k f(x)\,dx$. Thus $\sum a_n$ converges; that is, $\lim_{k\to\infty} s_k$ exists and is a nimber is equivalent to $\int_0^\infty f(x)\,dx$ converges. When each $a_n \geq 0$, $\sum a_n$ converges means that the area under the graph of $y = f(x)$ is finite.