Review for Final Exam.

- 10 or 14 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Sections 6.1, 6.3, 6.5.
  - Sections 7.1-7.7.
  - Sections 8.1-8.5, 8.7.
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Solving differential equations (7.4)

Example

Find the function y solution of $y' = \dfrac{\cos(x)}{y^2}$ and $y(0) = 1$.

Solution:

$$y^2\, y' = \cos(x) \quad \Rightarrow \quad \int y^2(x)\, y'(x)\, dx = \int \cos(x)\, dx.$$

The substitution $u = y(x)$, with $du = y'(x)\, dx$, implies

$$\int u^2\, du = \int \cos(x)\, dx \;\Rightarrow\; \frac{u^3}{3} = \sin(x) + c,$$

Therefore, $y^3(x) = 3(\sin(x) + c)$. Then, $y(x) = \sqrt[3]{3c + 3\sin(x)}$.
Furthermore,

$$1 = y(0) = \sqrt[3]{3c + 0} \;\Rightarrow\; 3c = 1.$$

We conclude that $y(x) = \sqrt[3]{1 + \sin(x)}$. $\triangleleft$

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Sections 8.1-8.5, 8.7.

Example

Evaluate $I = \displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)}\,dx.$

Solution: Substitution: $u = \tan(x)$, then $du = \sec^2(x)\,dx$.

$$I = \int_{u_0}^{u_1} \frac{du}{u} = \ln(u)\Big|_{u_0}^{u_1} = \ln(u_1) - \ln(u_0) = \ln\Big(\frac{u_1}{u_0}\Big)$$

where $u_1 = \tan(\pi/3)$, and $u_0 = \tan(\pi/4)$, that is,

$$u_1 = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}, \qquad u_2 = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1.$$

We conclude: $I = \ln\big(\sqrt{3}\big)$. ◁

Sections 8.1-8.5, 8.7.

Example

Evaluate $I = \displaystyle\int \frac{dx}{\sqrt{x^2 - 25}}$, for $x > 5$.

Solution: Substitution: $u = 5\sec(\theta)$, $du = 5\sec(\theta)\,\tan(\theta)\,d\theta$,

$$I = \int \frac{5\sec(\theta)\,\tan(\theta)}{5\sqrt{\sec^2(\theta) - 1}}\,d\theta = \int \frac{\sec(\theta)\,\tan(\theta)}{|\tan(\theta)|}\,d\theta.$$

Since $x > 5$, $\sec(\theta) > 0$ so $\tan(\theta) > 0$.

$$I = \int \frac{\sec(\theta)\,\tan(\theta)}{\tan(\theta)}\,d\theta = \int \sec(\theta)\,d\theta.$$

$$I = \int \sec(\theta)\,\frac{\big(\sec(\theta) + \tan(\theta)\big)}{\big(\sec(\theta) + \tan(\theta)\big)}\,d\theta.$$

Sections 8.1-8.5, 8.7.

Example

Evaluate $I = \int \frac{dx}{\sqrt{x^2 - 25}}$, for $x > 5$.

Solution: Recall: $I = \int \sec(\theta) \frac{\big(\sec(\theta) + \tan(\theta)\big)}{\big(\sec(\theta) + \tan(\theta)\big)}\, d\theta.$

$$I = \int \frac{\big(\sec^2(\theta) + \sec(\theta)\tan(\theta)\big)}{\big(\sec(\theta) + \tan(\theta)\big)}\, d\theta = \int \frac{\big(\tan'(\theta) + \sec'(\theta)\big)}{\big(\sec(\theta) + \tan(\theta)\big)}\, d\theta.$$

Substitute $u = \sec(\theta) + \tan(\theta)$, then $I = \int \frac{du}{u} = \ln(u) + c.$

$I = \ln\big(\sec(\theta) + \tan(\theta)\big) + c, \quad \sec(\theta) = \frac{x}{5}, \quad \tan^2(\theta) + 1 = \sec^2(\theta).$

$$I = \ln\Big(\frac{x}{5} + \sqrt{\frac{x^2}{5^2} - 1}\Big) + c. \qquad \triangleleft$$

Improper integrals (8.7): Comparison tests

Example

Determine whether $I = \int_3^\infty \frac{x\, dx}{\sqrt[3]{x^5 + x^3}}$ converges or not.

Solution: Limit comparison test. Let $g(x)$ such that:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x}{\sqrt[3]{x^5 + x^3}} = \lim_{x\to\infty} \frac{1}{x^{\frac{5}{3}-1}} = \lim_{x\to\infty} \frac{1}{x^{\frac{2}{3}}}.$$

Therefore, we use the limit comparison test with $g(x) = \frac{1}{x^{2/3}}$.

Then, by construction,

$$\lim_{x\to\infty} \Big(\frac{x}{\sqrt[3]{x^5 + x^3}}\Big)\big(x^{2/3}\big) = \lim_{x\to\infty} \Big(\frac{1}{x^{\frac{5}{3}-1}}\Big)\big(x^{2/3}\big) = 1.$$

Since $\int_3^\infty x^{-2/3}\, dx = 3\, x^{1/3}\Big|_3^\infty$ diverges, then I diverges too. $\triangleleft$

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Infinite series (10.2)

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2}{n}\right)^{\frac{3}{n}}$ converges or not.

Solution: The n-term test. We will need L'Hôpital's rule. Introduce the function $f(x)=\left(\frac{2}{x}\right)^{\frac{3}{x}}$.

$$\lim_{x\to\infty}\left(\frac{2}{x}\right)^{\frac{3}{x}}=\lim_{x\to\infty}e^{\left[\frac{3\ln\left(\frac{2}{x}\right)}{x}\right]}=e^{3\lim_{x\to\infty}\left[\frac{\ln\left(\frac{2}{x}\right)}{x}\right]}$$

L'Hôpital's rule to find the limit in the exponent;

$$\tilde{L}=\lim_{x\to\infty}\frac{\ln\left(\frac{2}{x}\right)}{x}=\lim_{x\to\infty}\frac{\left(\frac{x}{2}\,\frac{(-2)}{x^2}\right)}{1}=\lim_{x\to\infty}-\frac{1}{x}=0.$$

$\lim_{x\to\infty}f(x)=e^0=1$, then $\displaystyle\lim_{n\to\infty}\left(\frac{2}{n}\right)^{\frac{3}{n}}=1$. The series diverges.

Convergence tests for infinite series (10.5)

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)}{(n+1)\,e^n}$ converges or not.

Solution: The ratio test implies

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+2)\,e^{(n+1)}}\,\frac{(n+1)\,e^n}{n!} = \frac{1}{e}\,\frac{(n+2)}{(n+1)}\left[\frac{(n+1)\,n!}{n!}\right]$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)}{e} = \infty.$

Then, the ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)}{(n+1)\,e^n}$ diverges. $\triangleleft$

Power and Taylor series (10.7-10.9)

Example

Find the T_3 centered at $x=0$ of $f(x) = \dfrac{1}{1-x}$ and estimate the error of using T_3 to approximate f over $[-1/2, 1/2]$.

Solution: Recall geometric series: $T_3(x) = 1 + x + x^2 + x^3$.

A bound for the error on f by T_n centered at a over $[b,c]$ is

$$|R_n(x)| \leqslant M\frac{|x-a|^{n+1}}{(n+1)!}, \quad |f^{(n+1)}(x)| \leqslant M \quad \text{over} \quad [b,c].$$

Our case: $n=3$, $a=0$, $[b,c] = [-1/2, 1/2]$, $f^{(4)}(x) = \dfrac{4!}{(1-x)^5}$.

Since $|f^{(4)}(x)| \leqslant f^{(4)}(1/2) = \dfrac{4!}{(1/2)^5} = 2^5\,4! = M$, then

$$|R_3(x)| \leqslant \frac{2^5\,4!\,|x|^4}{4!} \;\Rightarrow\; |R_3(x)| \leqslant 2^5(1/2)^4 \;\Rightarrow\; |R_3(x)| \leqslant 2. \triangleleft$$

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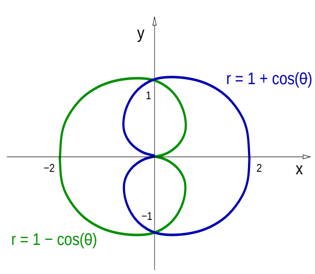
Parametric curves and polar coordinates (11.1-11.4)

Example

Compute the area shared by the interiors of the curves $r_1(\theta) = 1 + \cos(\theta)$ and $r_2(\theta) = 1 - \cos(\theta)$ for $\theta \in [0, 2\pi]$.

Solution: This is the intersection of two cardiods



By symmetry, the area of the interiors is

$$A = 4\int_0^{\pi/2} \frac{1}{2}\big(1 - \cos(\theta)\big)^2\, d\theta$$

$$A = 2\int_0^{\pi/2} \big(1 - 2\cos(\theta) + \cos^2(\theta)\big)\, d\theta$$

$$A = 2\Big[\frac{\pi}{2} - 2\sin(\theta)\Big|_0^{\pi/2} + 2\int_0^{\pi/2} \frac{1}{2}\big(1 + \cos(2\theta)\big)\, d\theta.$$

$$A = \pi - 4 + \frac{\pi}{2} + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2} \quad \Rightarrow \quad A = \frac{3\pi}{2} - 4. \qquad \triangleleft$$