Review for Final Exam.

- 10 or 14 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Sections 6.1, 6.3, 6.5.
  - Sections 7.1-7.7.
  - Sections 8.1-8.5, 8.7.
  - Sections 10.1-10.10.
  - Sections 11.1-11.5.

Review for Final Exam.

- 10 or 14 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - **Sections 6.1, 6.3, 6.5.**
  - Sections 7.1-7.7.
  - Sections 8.1-8.5, 8.7.
  - Sections 10.1-10.10.
  - Sections 11.1-11.5.

Volumes using cross-sections (6.1)

Example

Find the volume of the solid between the planes $x = 3$ and $x = -3$ with cross-sections perpendicular to the x-axis given by squares inscribed in the circle $x^2 + y^2 = 9$.

Solution: Let $a(x)$ denote the side of a cross-section perpendicular at the x-axis. The area of such section is

$$A(x) = \big[a(x)\big]^2, \qquad a(x) = 2\sqrt{9 - x^2}.$$

The volume of such solid is

$$V = \int_{-3}^{3} A(x)\,dx = \int_{-3}^{3} 4(9 - x^2)\,dx = 4\Big(9x - \frac{x^3}{3}\Big)\Big|_{-3}^{3}$$

Then, $V = 4[(9)(6) - (9)(2)]$, that is, $V = (9)(16)$. $\lhd$

Volumes using cross-sections (6.1)

Example

Find the volume of the region obtained by rotation the curve $x(y) = \tan(\pi y/8)$ for $y \in [0, 2]$ about the y-axis.

Solution:

To graph the function

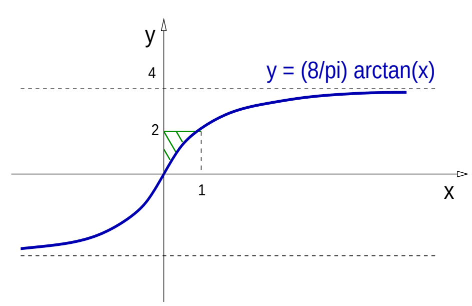
$$x = \tan(\pi y/8), \ y \in [0, 2],$$

one can graph

$$y = (8/\pi)\arctan(x).$$

Notice that

$$y \in [0, 2] \Rightarrow x \in [0, 1].$$



Therefore, $V = \pi \displaystyle\int_0^2 \big[x(y)\big]^2\,dy = \pi \int_0^2 \Big[\tan\Big(\frac{\pi y}{8}\Big)\Big]^2\,dy.$

Volumes using cross-sections (6.1)

Example

Find the volume of the region obtained by rotation the curve $x(y) = \tan(\pi y/8)$ for $y \in [0, 2]$ about the y-axis.

Solution: Recall: $V = \pi \int_0^2 \tan^2\Big(\frac{\pi y}{8}\Big)\, dy.$

Introduce the substitution $u = \pi y/8$, so $du = (\pi/8)\, dy$,

$$V = \pi \frac{8}{\pi} \int_0^{\pi/4} \tan^2(u)\, du = 8 \int_0^{\pi/4} \frac{[1 - \cos^2(u)]}{\cos^2(u)}\, du$$

$$V = 8 \int_0^{\pi/4} \Big[\frac{1}{\cos^2(u)} - 1\Big]\, du = 8 \int_0^{\pi/4} \big[\tan'(u) - 1\big]\, du.$$

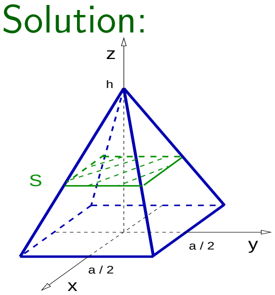
$$V = 8\big[\tan(u) - u\big]\Big|_0^{\pi/4} \quad \Rightarrow \quad V = 8\Big(1 - \frac{\pi}{4}\Big). \qquad \triangleleft$$

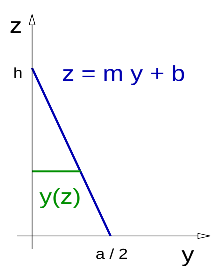
Volumes integrating cross-sections: General case.

Example

Find the volume of a pyramid with square base side a and height h.

Solution:





$$A(z) = \big[2y(z)\big]^2$$

We must find and invert

$$z(y) = my + b.$$

$$h = z(0) = b, \quad 0 = z(a/2) = m\frac{a}{2} + h \quad \Rightarrow \quad m = -\frac{2h}{a}.$$

$$z(y) = -\frac{2h}{a}\, y + h \quad \Rightarrow \quad y(z) = -\frac{a}{2h}\,(z - h).$$

$$V = \int_0^h \Big[-2\frac{a}{2h}\,(z - h)\Big]^2 dz = \frac{a^2}{h^2}\Big[\frac{(z-h)^3}{3}\Big|_0^h\Big] \Rightarrow V = \frac{1}{3} a^2 h. \qquad \triangleleft$$

Arc-length of curves on the plane (6.3, 11.2)

Remark: Curves on the xy-plane:

- A curve can be given as the graph of a function,

$$y = f(x), \qquad x \in [x_0, x_1].$$

- The length of the curve in this case is, (6.3),

$$L = \int_{x_0}^{x_1} \sqrt{1 + \big[y'(x)\big]^2}\, dx.$$

- Or a curve can be given in parametric form, as the set of points $(x(t), y(t))$ for $t \in [t_0, t_1]$.
- The length of the curve in this case is, (11.2)

$$L = \int_{t_0}^{t_1} \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\, dt.$$

Arc-length of curves on the plane (6.3)

Example

Find the arc-length of the function $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, for $x \in [1, 3]$.

Solution: Recall: $L = \displaystyle\int_{x_0}^{x_1} \sqrt{1 + \big[y'(x)\big]^2}\, dx$. Find y',

$$y'(x) = x^2 - \frac{1}{4x^2} \quad \Rightarrow \quad 1 + \big[y'(x)\big]^2 = 1 + x^4 + \frac{1}{16x^4} - \frac{1}{2},$$

$$1 + \big[y'(x)\big]^2 = x^4 + \frac{1}{16x^4} + \frac{1}{2} = \Big(x^2 + \frac{1}{4x^2}\Big)^2.$$

$$L = \int_1^3 \Big(x^2 + \frac{1}{4x^2}\Big)\, dx = \Big(\frac{x^3}{3} - \frac{1}{4x}\Big)\Big|_1^3 = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}.$$

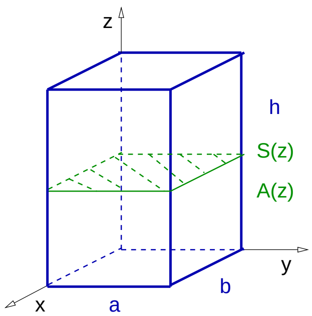
We conclude that $L = 9 - 1/6$. $\triangleleft$

Work and fluid forces: Pumping liquids

Example

A rectangular container with sides a, b, and height h, is filled with kerosene weighing $k = 51.3$ lb per cubic ft. Find the work needed to empty the container if the kerosene is pumped out from the top of the tank.

Solution:



The force is the kerosene weight:

$$F = k\,A(z)\,dz = k\,(ab)\,dz$$

The work done to lift that liquid from a height z to h is

$$W(z) = k\,(ab)(h - z)\,dz.$$

To empty the container: $W = k\,(ab)\int_0^h (h - z)\,dz = k\,(ab)\frac{h^2}{2}.$

Review for Final Exam.

- 10 or 14 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers:
  - Sections 6.1, 6.3, 6.5.
  - **Sections 7.1-7.7.**
  - Sections 8.1-8.5, 8.7.
  - Sections 10.1-10.10.
  - Sections 11.1-11.5.

The inverse function (7.1).

Example

Find the inverse of $f(x) = 6x^2 - 24x + 24$ for $x \leqslant 2$.

Solution: We call $y = f(x)$, and we find $x(y)$.

$$y = 6(x^2 - 4 + 4) = 6(x-2)^2 \quad \Rightarrow \quad (x-2)^2 = \frac{y}{6}$$

$$x_\pm - 2 = \pm\sqrt{\frac{y}{6}} \quad \Rightarrow \quad x_\pm = 2 \pm \sqrt{\frac{y}{6}}.$$

Since we are interested in the inverse for $x \leqslant 2$,

$$x = 2 - \sqrt{\frac{y}{6}}.$$

We can rewrite the answer as $f^{-1}(y) = 2 - \sqrt{y/6}$.

It is also correct to write $f^{-1}(x) = 2 - \sqrt{x/6}$. $\triangleleft$

The inverse function (7.1).

Example

Given $f(x) = 2x^3 + 3x^2 + 3$ for $x \geqslant 0$, find $\dfrac{df^{-1}}{dx}$ at $x = 8$.

Solution: We use y for the variable of f^{-1}. Find $\dfrac{df^{-1}}{dy}$ at $y = 8$.

Recall the main formula: $(f^{-1})'(y=8) = \dfrac{1}{f'(f^{-1}(y=8))}$.

We need to find $x = f^{-1}(y = 8)$. Since

$$8 = y = f(x) = 2x^3 + 3x^2 + 3,$$

by trial an error, $x = 1$. So, $f(x=1) = 8$ and $f^{-1}(y=8) = 1$.

$$(f^{-1})'(y=8) = \frac{1}{f'(f^{-1}(y=8))} = \frac{1}{f'(x=1)}.$$

We need $f'(x=1)$. But $f'(x) = 6x^2 + 6x \;\Rightarrow\; f'(x=1) = 12$.

We obtain $(f^{-1})'(8) = 1/12$. $\triangleleft$

The natural logarithm (7.2)

Example

Evaluate $I = \int_1^{e^\pi} \sin\big(\ln(x)\big)\, dx.$

Solution: We try the substitution $u = \ln(x)$, hence $du = dx/x$.

Notice $dx = x\, du = e^u\, du$. Therefore, $I = \int_0^{\pi} e^u \sin(u)\, du.$

Integrate by parts twice, first, $f = e^u$, $g' = \sin(u)$

$$\int e^u \sin(u)\, du = -e^u \cos(u) + \int e^u \cos(u)\, du$$

$$\int e^u \sin(u)\, du = -e^u \cos(u) + e^u \sin(u) - \int e^u \sin(u)\, du.$$

$$\int e^u \sin(u)\, du = \frac{1}{2}\, e^u \big(\sin(u) - \cos(u)\big).$$

So, $I = \frac{1}{2}\, e^u \big(\sin(u) - \cos(u)\big)\Big|_0^{\pi}$, hence $I = \frac{1}{2}(e^\pi - 1)$. ◁

The natural logarithm (7.2)

Example

Simplify $f(x) = \ln\Big(\frac{\sin^5(2t)}{7}\Big)$, and find the derivatives of $g(x) = 3\ln(6\ln(x))$, and $h(x) = \ln\big(\sqrt{25\sin(x)\cos(x)}\big)$.

Solution: First: $f(x) = \ln(\sin^5(2t)) - \ln(7)$,

so we conclude that $f(x) = 5\ln(\sin(2t)) - \ln(7)$.

Second, $g'(x) = 3\,\frac{1}{6\ln(x)}\,(6\ln(x))'$, that is, $g'(x) = 3\,\frac{1}{\ln(x)}\,\frac{1}{x}$.

Sometimes it is better simplify first and derivate later,

$$h(x) = \frac{1}{2}\big[\ln(25) + \ln(\sin(x)) + \ln(\cos(x)\big],$$

$$h'(x) = \frac{1}{2}\Big[\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}\Big].$$ ◁

The natural logarithm (7.2)

Example

Find $I = \displaystyle\int \frac{\sec(x)}{\sqrt{\ln(\sec(x)+\tan(x))}}\,dx.$

Solution: We try the substitution $u = \ln(\sec(x)+\tan(x))$. Recall

$$\sec(x)+\tan(x) = \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = \frac{1+\sin(x)}{\cos(x)},$$

$$du = \frac{\cos(x)}{1+\sin(x)}\left[\frac{\cos(x)\cos(x) - (1+\sin(x))(-\sin(x))}{\cos^2(x)}\right] dx$$

$$du = \frac{\cos(x)}{[1+\sin(x)]}\,\frac{[1+\sin(x)]}{\cos^2(x)}\,dx = \frac{1}{\cos(x)}\,dx = \sec(x)\,dx.$$

$$I = \int \frac{du}{u^{1/2}} = 2\,u^{1/2} \Rightarrow I = 2\sqrt{\ln(\sec(x)+\tan(x))}. \qquad \triangleleft$$