## Review for Exam 3.

- 5 or 6 problems.
- Exam covers: 10.2-10.10, 11.1-11.5.
  - Infinite series (10.2).
  - The integral test (10.3).
  - Comparison tests (10.4).
  - The ratio test (10.5).
  - Alternating series (10.6).
  - Power series (10.7).
  - Taylor and Maclaurin series (10.8).
  - Convergence of Taylor series (10.9).
  - The binomial series (10.10).
  - Parametrization of plane curves (11.1).
  - Calculus with parametric curves (11.2).
  - Polar coordinates (11.3).
  - Graphing in polar coordinates (11.4).
- Areas in polar coordinates (11.5), not included.

## Convergence tests for infinite series (10.2)

Example

Determine whether the series below converge or not. Specify the test you use: (a) $\displaystyle\sum_{n=1}^{\infty} \ln\Big(\frac{n}{2n+2}\Big)$; (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}-1}{7^n}$.

Solution: Recall: The simplest test to do is the $n$-th term test:
If $a_n \to L \neq 0$, then the series $\sum a_n$ diverges.

(a) $$\lim_{n\to\infty} \ln\Big(\frac{n}{2n+2}\Big) = \lim_{n\to\infty} \ln\Big(\frac{n}{2n}\Big) = \ln\Big(\frac{1}{2}\Big) \neq 0.$$

The series $\displaystyle\sum_{n=1}^{\infty} \ln\Big(\frac{n}{2n+2}\Big)$ diverges.

(b) $$\lim_{n\to\infty} \frac{2^{n-1}-1}{7^n} = \lim_{n\to\infty} \frac{2^{n-1}}{7^n} - \frac{1}{7^n} = \lim_{n\to\infty} \frac{1}{2}\Big(\frac{2}{7}\Big)^n - \Big(\frac{1}{7}\Big)^n = 0.$$

The series may converge or diverge. It converges: Geometric series.

## Convergence tests for infinite series (10.2)

### Example

Determine whether the series below converge or not. Specify the test you use: (a) $\displaystyle\sum_{n=1}^{\infty} \ln\Big(\frac{n}{2n+2}\Big)$; (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}-1}{7^n}$.

Solution: The series in (b) is a Geometric series. Indeed,

$$S = \sum_{n=0}^{\infty} \frac{2^{n-1}-1}{7^n} = \sum_{n=0}^{\infty} \Big(\frac{2^{n-1}}{7^n} - \frac{1}{7^n}\Big) = \frac{1}{2}\sum_{n=0}^{\infty} \frac{2^n}{7^n} - \sum_{n=0}^{\infty} \frac{1}{7^n}.$$

$$S = \frac{1}{2}\sum_{n=0}^{\infty} \Big(\frac{2}{7}\Big)^n - \sum_{n=0}^{\infty} \Big(\frac{1}{7}\Big)^n = \frac{1}{2}\,\frac{1}{\big(1-\frac{2}{7}\big)} - \frac{1}{\big(1-\frac{1}{7}\big)}$$

We conclude that $\displaystyle\sum_{n=0}^{\infty} \frac{2^{n-1}-1}{7^n} = \frac{7}{10} - \frac{7}{6}$. $\triangleleft$

## Convergence tests for infinite series (10.3)

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{2}{n\sqrt{3n}}\Big)$ converges or not. Specify the test you use.

Solution: Notice: $n$-th term test trivially gives $\displaystyle\lim_{n\to\infty} \frac{2}{n\sqrt{3n}} = 0$.

$n$-term test inconclusive. Notice that $f(x) = \dfrac{2}{x\sqrt{3x}}$ is integrable in the interval $[1,\infty)$. Indeed,

$$\int_1^{\infty} \frac{2}{x\sqrt{3x}}\,dx = \frac{2}{\sqrt{3}}\int_1^{\infty} x^{-3/2}\,dx = \frac{2}{\sqrt{3}}\big(-2x^{-1/2}\big)\Big|_1^{\infty} = \frac{4}{\sqrt{3}}.$$

Therefore, the integral test implies that $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{2}{n\sqrt{3n}}\Big)$ converges. $\triangleleft$

## Convergence tests for infinite series (10.4)

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n^2+8}}$ converges or not.

Specify the test you use.

Solution: Notice: $n$-th term test gives $\displaystyle\lim_{n\to\infty} \frac{5}{n\sqrt{n^2+8}} = 0.$

$n$-term test inconclusive. However we can compare the series with an $\sum(1/n^2)$ series, which is convergent.

$$n^2 < n^2 + 8 \quad \Rightarrow \quad \frac{1}{n^2+8} < \frac{1}{n^2} \quad \Rightarrow \quad \frac{1}{\sqrt{n^2+8}} < \frac{1}{n}$$

$$\frac{1}{n\sqrt{n^2+8}} < \frac{1}{n^2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n^2+8}} < \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Since the series on the right-hand side converges (integral test)

Then comparison test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n\sqrt{n^2+8}}$ converges. $\lhd$

## Convergence tests for infinite series (10.5)

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n^2+2)}{e^{6n}}$ converges or not.

Specify the test you use.

Solution: Notice: $n$-th term test gives $\displaystyle\lim_{n\to\infty} \frac{(n^2+2)}{e^{6n}} = 0.$

$n$-term test inconclusive. However, ratio test implies

$$\frac{a_{n+1}}{a_n} = \frac{[(n+1)^2+2]}{e^{6(n+1)}} \frac{e^{6n}}{(n^2+2)} = \frac{1}{e^6} \frac{[(n^2+2n+3]}{(n^2+2)}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e^6} < 1.$

Then, the ratio test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{(n^2+2)}{e^{6n}}$ converges. $\lhd$

## Convergence tests for infinite series (10.6)

### Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n+2)!}{(-3)^n n!}$ converges or not.

Specify the test you use.

Solution: Notice: $n$-th term test gives $\displaystyle\lim_{n\to\infty} \frac{(n+2)!}{(-3)^n n!} = 0.$

$n$-term test inconclusive. This is an alternating series. We use the ratio test on $|a_n|$, that is,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+3)!}{3^{n+1}(n+1)!}\,\frac{3^n n!}{(n+2)!} = \frac{1}{3}\,\frac{(n+3)!}{(n+2)!}\,\frac{n!}{(n+1)!}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3}\,\frac{(n+3)}{(n+1)} \quad\Rightarrow\quad \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{3} < 1.$$

Then ratio test implies the series above converges absolutely; and the alternating series theorem implies that it converges. ◁

## Power and Taylor series (10.7-10.9)

### Example

Find the Taylor polynomial order 3 centered at $x = 0$ of the function $f(x) = e^{-2x}$. Estimate the error made when using this polynomial to approximate $f$ over $[-2, 2]$.

Solution: The Taylor polynomial is $\displaystyle T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}\,x^n.$

$$f'(x) = -2\,e^{-2x}, \quad f''(x) = 4\,e^{-2x}, \quad f'(x) = -8\,e^{-2x},$$

$$T_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3$$

$$T_3(x) = 1 - 2\,x + \frac{4}{2!}\,x^2 - \frac{8}{3!}\,x^3 \Rightarrow T_3(x) = 1 - 2\,x + 2\,x^2 - \frac{4}{3}\,x^3.$$

## Power and Taylor series (10.7-10.9)

Example

Find the Taylor polynomial order 3 centered at $x = 0$ of the function $f(x) = e^{-2x}$. Estimate the error made when using this polynomial to approximate $f$ over $[-2, 2]$.

Solution: Recall: $T_3(x) = 1 - 2\,x + 2\,x^2 - \frac{4}{3}\,x^3.$

A bound for the error on $f$ by $T_n$ centered at $a$ over $[b, c]$ is

$$|R_n(x)| \leqslant M \frac{|x-a|^{n+1}}{(n+1)!}, \quad |f^{(n+1)}(x)| \leqslant M \quad \text{over} \quad [b, c].$$

In our case: $n = 3$, $a = 0$, $[b, c] = [-2, 2]$ and $f^{(4)}(x) = 16\,e^{-2x}$.

Since $|f^{(4)}(x)| \leqslant f^{(4)}(-2) = 16\,e^4 = M$, then

$$|R_3(x)| \leqslant \frac{16\,e^4 |x|^4}{4!} \quad \Rightarrow \quad |R_3(x)| \leqslant \frac{2\,e^4\,2^4}{3}.$$

$\lhd$

## Power and Taylor series (10.7-10.9)

Example

Find the Taylor series centered at $x = 0$ of $f(x) = e^{-2x}$ and determine the open interval of convergence.

Solution:

Recall: $f'(x) = -2\,e^{-2x}$, $f''(x) = 4\,e^{-2x}$, $f'(x) = -8\,e^{-2x}$.

This implies the formula $f^{(n)}(x) = (-2)^n\,e^{-2x}$. Therefore,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\,x^n \quad \Rightarrow \quad f(x) = \sum_{n=0}^{\infty} \frac{(-2)^n}{n!}\,x^n$$

The interval of convergence can be obtained with the ratio test,

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{(-2)^{n+1}}{(n+1)!}\,x^{n+1}\frac{n!}{(-2)^n x^n}\right| = 2|x|\frac{1}{n+1}.$$

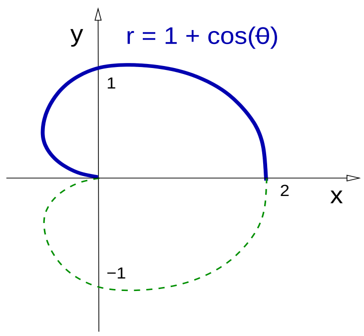
Since $\lim_{n\to\infty} \frac{2|x|}{n+1} = 0$, the interval of convergence is $(-\infty, \infty)$. $\lhd$

# Parametric curves and polar coordinates (11.1-11.4)

Example

Graph on the $xy$-plane the curve $r(\theta) = 1 + \cos(\theta)$ for $\theta \in [0, \pi]$. Find the slope of the line tangent to the curve at $\theta = \pi/2$.

Solution: This is half a cardiod:




The equation for the tangent line slope at $\theta$ is $m = \dfrac{y'(\theta)}{x'(\theta)}$.

$$y(\theta) = r(\theta)\,\sin(\theta),$$

$$x(\theta) = r(\theta)\,\cos(\theta).$$

$$y'(\theta) = [-\sin(\theta)]\,\sin(\theta) + [1 + \cos(\theta)]\cos(\theta) \;\Rightarrow\; y'(\pi/2) = -1.$$

$$x'(\theta) = [-\sin(\theta)]\,\cos(\theta) + [1 + \cos(\theta)][-\sin(\theta)] \;\Rightarrow\; x'(\pi/2) = -1.$$

We conclude: $m = 1$. $\triangleleft$