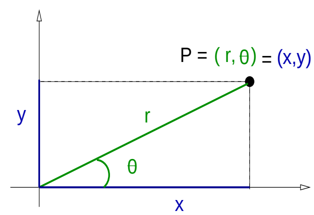
Area of regions in polar coordinates (Sect. 11.5)

- Review: Few curves in polar coordinates.
- Formula for the area or regions in polar coordinates.
- Calculating areas in polar coordinates.

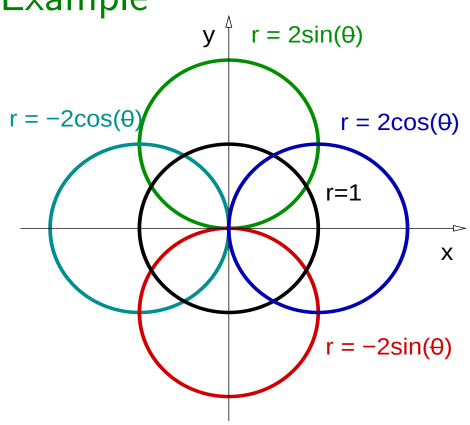
Transformation rules Polar-Cartesian.

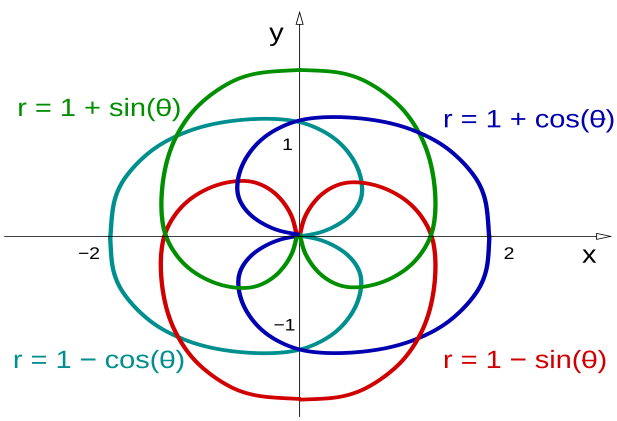
Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ), with $r > 0$ and $\theta \in [0, 2\pi)$ defined by the picture.



Example





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Formula for the area or regions in polar coordinates

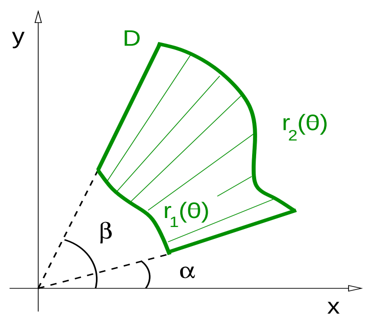
Theorem

If the functions $r_1, r_2 : [\alpha, \beta] \to \mathbb{R}$ are continuous and $0 \leqslant r_1 \leqslant r_2$, then the area of a region $D \subset \mathbb{R}^2$ given by

$$D = \big\{(r, \theta) \in \mathbb{R}^2 \ : \ r \in [r_1(\theta), r_2(\theta)], \ \theta \in [\alpha, \beta]\big\}.$$

is given by the integral

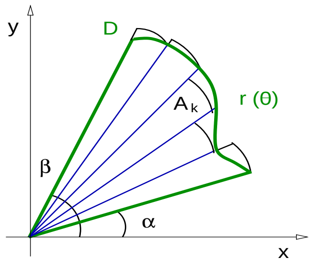
$$A(D) = \int_\alpha^\beta \frac{1}{2} \Big(\big[r_2(\theta)\big]^2 - \big[r_1(\theta)\big]^2 \Big)\, d\theta.$$



Remark: This result includes the case of $r_1 = 0$, which are fan-shaped regions.

Formula for the area or regions in polar coordinates

Idea of the Proof: Introduce a partition $\theta_k = k\,\Delta\theta$, with $k = 1, \cdots, n$, and $\Delta\theta = \dfrac{\beta - \alpha}{n}$



The area of each fan-shaped region on the figure is,

$$A_k = \frac{1}{2}\big[r(\theta_k)\big]^2\,\Delta\theta.$$

A Riemann sum that approximates the green region area is

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\big[r(\theta_k)\big]^2\,\Delta\theta.$$

Refining the partition and taking a limit $n \to \infty$ one can prove that the Riemann sum above converges and the limit is called

$$A(D) = \int_{\alpha}^{\beta} \frac{1}{2}\big[r(\theta)\big]^2\,d\theta.$$

□

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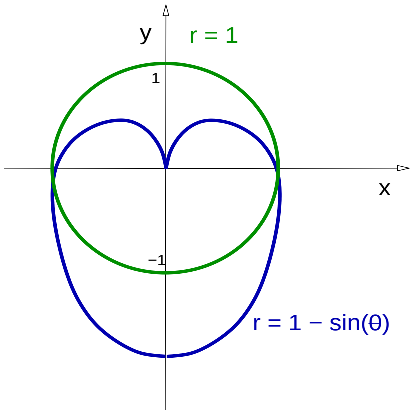
Calculating areas in polar coordinates

Example

Find the area inside the circle $r = 1$ and outside the cardiod $r = 1 - \sin(\theta)$.

Solution:



The Theorem implies

$$A = \int_{\alpha}^{\beta} \frac{1}{2} \left(1 - \left[1 - \sin(\theta)\right]^2\right) d\theta.$$

We need to find α and β. They are the intersection of the circle and the cardiod:

$$1 = 1 - \sin(\theta) \quad \Rightarrow \quad \sin(\theta) = 0 \quad \Rightarrow \quad \begin{cases} \alpha = 0, \\ \beta = \pi. \end{cases}$$

Calculating areas in polar coordinates

Example

Find the area inside the circle $r = 1$ and outside the cardiod $r = 1 - \sin(\theta)$.

Solution: Therefore: $A = \int_0^{\pi} \frac{1}{2} \left(1 - \left[1 - \sin(\theta)\right]^2\right) d\theta.$

$$A = \frac{1}{2} \int_0^{\pi} \left(2\sin(\theta) - \sin^2(\theta)\right) d\theta$$

$$A = \frac{1}{2} \int_0^{\pi} \left(2\sin(\theta) - \frac{1}{2}\left[1 - \cos(2\theta)\right]\right) d\theta$$

$$A = \frac{1}{2}\left(-2\cos(\theta)\Big|_0^{\pi} - \frac{1}{2}\left[\pi - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi}\right]\right)$$

$$A = \frac{1}{2}\left(4 - \frac{\pi}{2}\right) \quad \Rightarrow \quad A = 2 - \frac{\pi}{4}. \qquad \triangleleft$$

Calculating areas in polar coordinates

Example

Find the area of the intersection of the interior of the regions bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: We first review that these curves are actually circles.

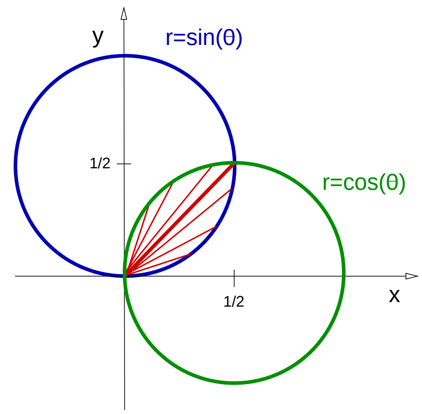
$$r = \cos(\theta) \quad \Leftrightarrow \quad r^2 = r\cos(\theta) \quad \Leftrightarrow \quad x^2 + y^2 = x.$$

Completing the square in x we obtain

$$\Big(x - \frac{1}{2}\Big)^2 + y^2 = \Big(\frac{1}{2}\Big)^2.$$

Analogously, $r = \sin(\theta)$ is the circle

$$x^2 + \Big(y - \frac{1}{2}\Big)^2 = \Big(\frac{1}{2}\Big)^2.$$



Calculating areas in polar coordinates

Example

Find the area of the intersection of the interior of the regions bounded by the curves $r = \cos(\theta)$ and $r = \sin(\theta)$.

Solution: The Theorem implies: $A = 2\int_0^{\pi/4} \frac{1}{2}\sin^2(\theta)\,d\theta$;

$$A = \int_0^{\pi/4} \frac{1}{2}\big[1 - \cos(2\theta)\big]\,d\theta = \frac{1}{2}\Big[\Big(\frac{\pi}{4} - 0\Big) - \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/4}\Big];$$

$$A = \frac{1}{2}\Big[\frac{\pi}{4} - \Big(\frac{1}{2} - 0\Big)\Big] = \frac{\pi}{8} - \frac{1}{4} \quad \Rightarrow \quad A = \frac{1}{8}(\pi - 2).$$

◁

Also works: $A = \int_0^{\pi/4} \frac{1}{2}\sin^2(\theta)\,d\theta + \int_{\pi/4}^{\pi/2} \frac{1}{2}\cos^2(\theta)\,d\theta.$