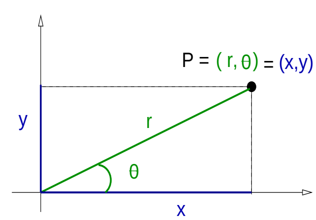
Graphing in polar coordinates (Sect. 11.4)

- Review: Polar coordinates.
- Review: Transforming back to Cartesian.
- Computing the slope of tangent lines.
- Using symmetry to graph curves.
- Examples:
  - Circles in polar coordinates.
  - Graphing the Cardiod.
  - Graphing the Lemniscate.

Transformation rules Polar-Cartesian.

Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ), with $r \geqslant 0$ and $\theta \in [0, 2\pi)$ defined by the picture.



Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point $P = (r, \theta)$ are given by

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ in the first and fourth quadrants are given by

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

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Review: Transforming back to Cartesian

Example

Find the equation of the curve in Cartesian coordinates for $r = 4\cos(\theta)$, for $\theta \in [-\pi/2, \pi/2]$.

Solution: Multiply by r the whole equation, $r^2 = 4r\cos(\theta)$.

Recall: $x = r\cos(\theta)$, and $y = r\sin(\theta)$, therefore $x^2 + y^2 = r^2$,

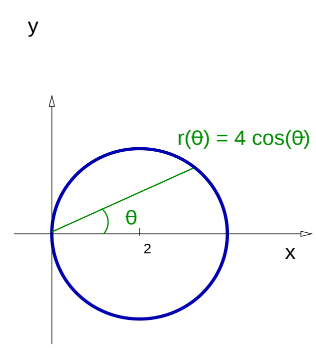
$$x^2 + y^2 = 4x \quad \Rightarrow \quad x^2 - 4x + y^2 = 0.$$

Complete the square:

$$\left[x^2 - 2\left(\frac{4}{2}\right)x + 4\right] - 4 + y^2 = 0$$

$$(x-2)^2 + y^2 = 4.$$

This is the equation of a circle radius $r = 2$ with center at $(2, 0)$. ◁



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Computing the slope of tangent lines

Recall: The slope of the line tangent to the curve $y = f(x)$, can be written in terms of $(x(t), y(t))$ as follows

$$\frac{df}{dx} = \frac{dy/dt}{dx/dt}.$$

Remark: If the curve is given in polar coordinates, $r = r(\theta)$, then

$$x(\theta) = r(\theta)\,\cos(\theta) \qquad y(\theta) = r(\theta)\,\sin(\theta).$$

The formula for the slope is then

$$\frac{df}{dx} = \frac{y'(\theta)}{x'(\theta)} \quad \Rightarrow \quad \frac{df}{dx} = \frac{r'(\theta)\sin(\theta) + r(\theta)\,\cos(\theta)}{r'(\theta)\cos(\theta) - r(\theta)\,\sin(\theta)}.$$

If the curve passes through the origin, $r(\theta_0) = 0$, then

$$\left.\frac{df}{dx}\right|_{\theta_0} = \frac{r'(\theta_0)\sin(\theta_0)}{r'(\theta_0)\cos(\theta_0)}.$$

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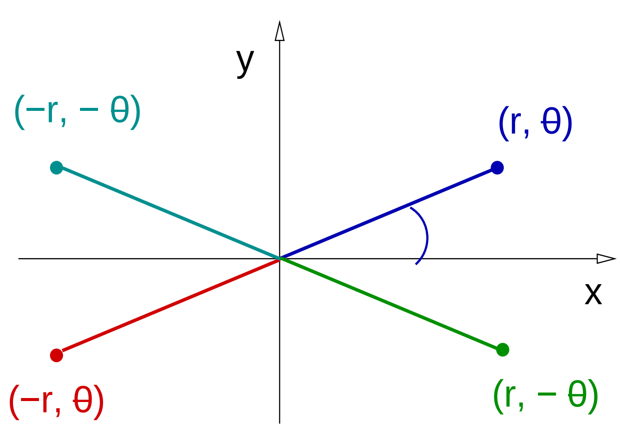
Using symmetry to graph curves

Remark: If a curve is symmetric under reflections about the *x-axis*, or the *y-axis*, or the *origin*, then the work needed to graph of the curve can be reduced.

- *x-axis symmetry:* (r, θ) and $(r, -\theta)$ belong to the graph.
- *Origin symmetry:* (r, θ) and $(-r, \theta)$ belong to the graph.
- *y-axis symmetry:* (r, θ) and $(-r, -\theta)$ belong to the graph.



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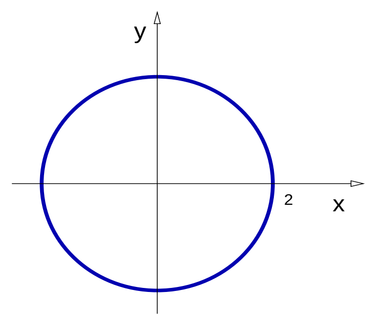
Circles in polar coordinates

Remark: Circles centered at the origin are trivial to graph.



Example

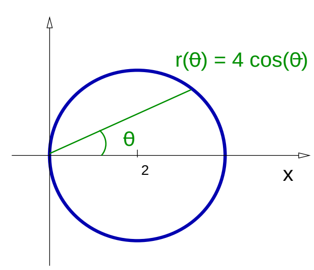
Graph the curve $r = 2$, $\theta \in [0, 2\pi)$.

Remark:

Circles not centered at the origin are more complicated to graph.

Example

Graph the curve
$r = 4\cos(\theta)$, $\theta \in [0, 2\pi)$.



Solution: Back to Cartesian:

Circles in polar coordinates

Remark: We now use the graph of the function $r = 4\cos(\theta)$ to graph the curve $r = 4\cos(\theta)$ in the xy-plane.

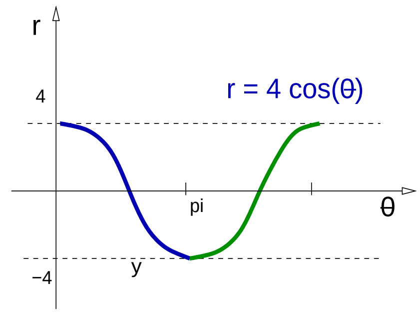
Example

Graph the curve $r = 4\cos(\theta), \quad \theta \in [0, 2\pi)$.

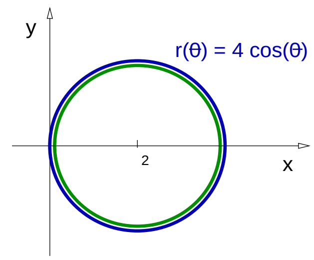
Solution:

Notice that $r(\theta) = r(-\theta)$.
(Reflection about x-axis symmetry.)

The graph of $r = 4\cos(\theta)$ is



The graph above helps to do the curve on the xy-plane. We actually cover the circle twice!



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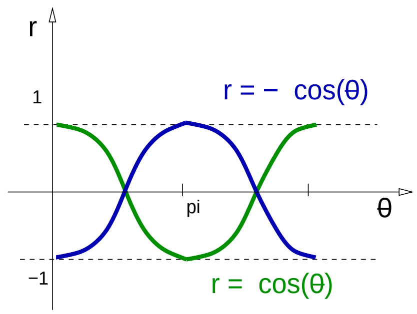
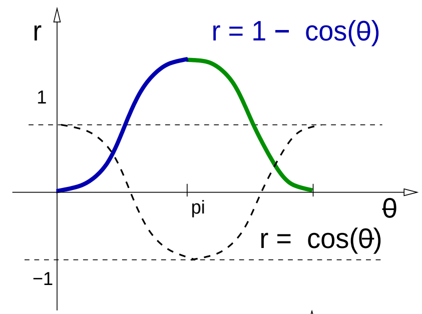
Graphing the Cardiod

Example

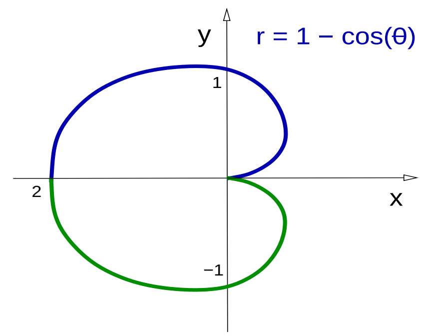
Graph on the *xy*-plane the curve $r = 1 - \cos(\theta)$, $\theta \in [0, 2\pi)$.

Solution: We first graph the function $r = 1 - \cos(\theta)$.





From the previous graph we obtain the curve: on the *xy*-plane:

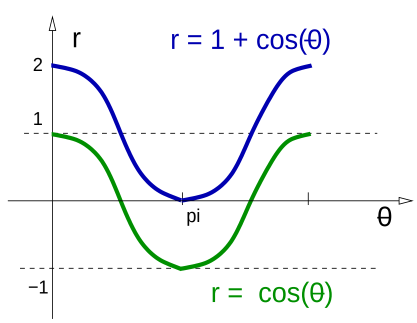


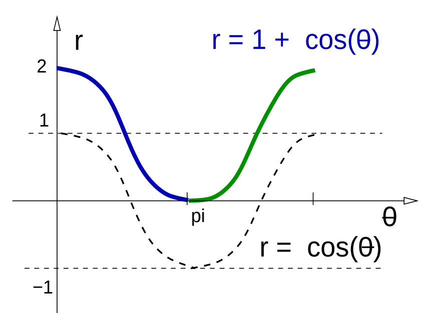
Graphing the Cardiod

Example

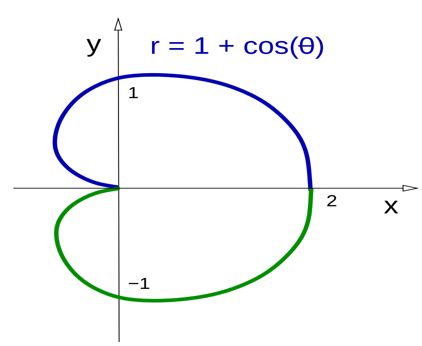
Graph on the *xy*-plane the curve $r = 1 + \cos(\theta)$, $\theta \in [0, 2\pi)$.

Solution: We first graph the function $r = 1 + \cos(\theta)$.





From the previous graph we obtain the curve: on the *xy*-plane:



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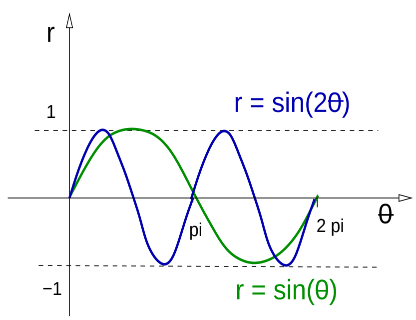
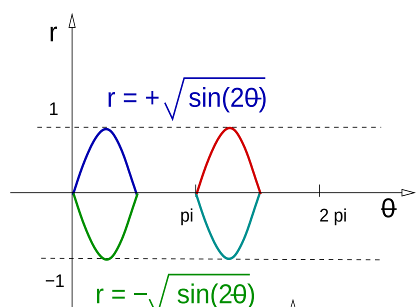
Graphing the Lemniscate

Example

Graph on the *xy*-plane the curve $r^2 = \sin(2\theta)$, $\theta \in [0, 2\pi)$.

Solution: We first graph the function $r = \pm\sqrt{\sin(2\theta)}$.





From the previous graph we obtain the curve: on the *xy*-plane: