Polar coordinates (Sect. 11.3)

- Review: Arc-length of a curve.
- Polar coordinates definition.
- Transformation rules Polar-Cartesian.
- Examples of curves in polar coordinates.

Review: Arc-length of a curve

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

Remark: If the curve $y = f(x)$ can be described by the parametric functions $\big(x(t), y(t)\big)$, for $t \in I \subset \mathbb{R}$, and if $x'(t) \neq 0$ for $t \in I$, then holds

$$\frac{df}{dx} = \frac{(dy/dt)}{(dx/dt)}.$$

Remark: The arc-length of a continuously differentiable curve $\big(x(t), y(y)\big)$, for $t \in [a, b]$ is the number

$$L = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\, dt.$$

Review: Arc-length of a curve

Remark:

- The formula for the arc-length

$$L = \int_a^b \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\,dt.$$

  can be used in a curve of the form $y = f(x)$.
- Just choose the trivial parametrization:

$$x(t) = t, \qquad y(t) = f(t).$$

- Then $x'(t) = 1$, $y'(t) = f'(t)$, and the arc-length formula is

$$L = \int_a^b \sqrt{1 + \big[f'(t)\big]^2}\,dt.$$

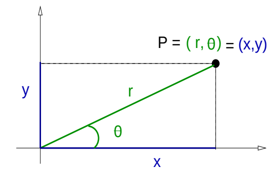
Polar coordinates (Sect. 11.3)

- Review: Arc-length of a curve.
- **Polar coordinates definition.**
- Transformation rules Polar-Cartesian.
- Examples of curves in polar coordinates.

Polar coordinates definition

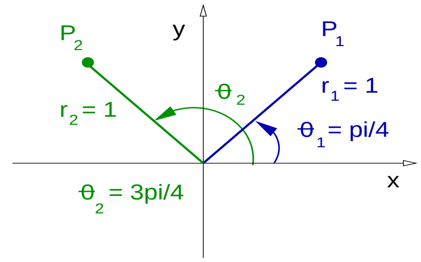
Definition

The *polar coordinates* of a point $P \in \mathbb{R}^2$ is the ordered pair (r, θ), with $r \geqslant 0$ and $\theta \in [0, 2\pi)$ defined by the picture.



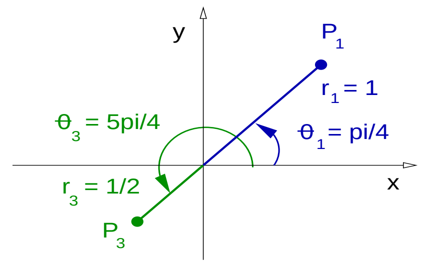
Example

Graph the points $P_1 = (1, \pi/4)$, $P_2 = (1, 3\pi/4)$.



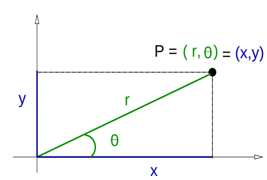
Example

Graph the points $P_1 = (1, \pi/4)$, $P_3 = (1/2, 5\pi/4)$.



Polar coordinates definition

Remark: The *polar coordinates* (r, θ) are restricted to $r \geqslant 0$ and $\theta \in [0, 2\pi)$.

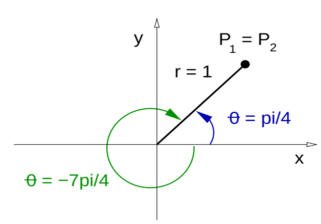


Remark:

- This restriction implies that for every point $P \neq (0,0)$ there is a unique pair (r, θ) to label that point.
- Usually this restriction is not applied, and $r \in \mathbb{R}$, $\theta \in \mathbb{R}$.
- This means that infinitely many ordered pairs (r, θ) label the same point P.

Example

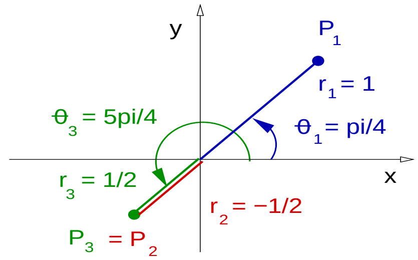
Graph the points $P_1 = (1, \pi/4)$ and $P_2 = (1, -7\pi/4)$.



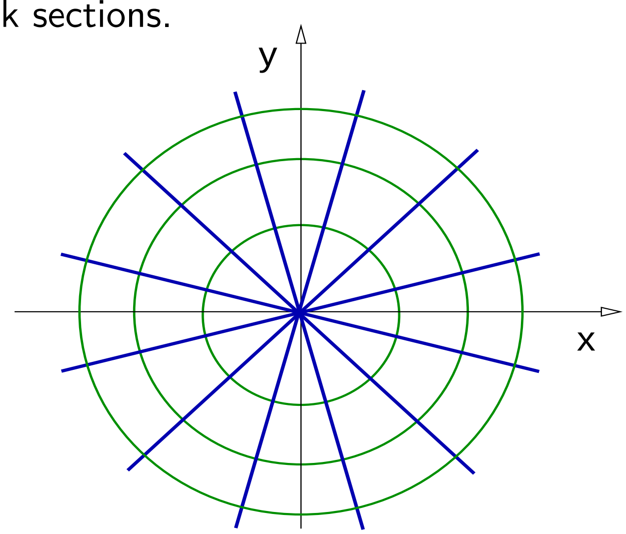
Polar coordinates definition

Example

Graph the points $P_1 = (1, \pi/4)$, $P_2 = (-1/2, \pi/4)$, and $P_3 = (1/2, 5\pi/4)$.



Remark: Polar coordinates are well adapted to describe circular curves and disk sections.

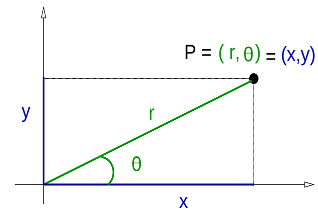


Polar coordinates (Sect. 11.3)

- Review: Arc-length of a curve.
- Polar coordinates definition.
- **Transformation rules Polar-Cartesian.**
- Examples of curves in polar coordinates.

Transformation rules Polar-Cartesian.

Remark: The *polar coordinates* (r, θ), with $r \geqslant 0$ and $\theta \in (-\pi, \pi]$ can be related to Cartesian coordinates.



Theorem (Cartesian-polar transformations)

The Cartesian coordinates of a point $P = (r, \theta)$ *are given by*

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

The polar coordinates of a point $P = (x, y)$ *in the first and fourth quadrants are given by*

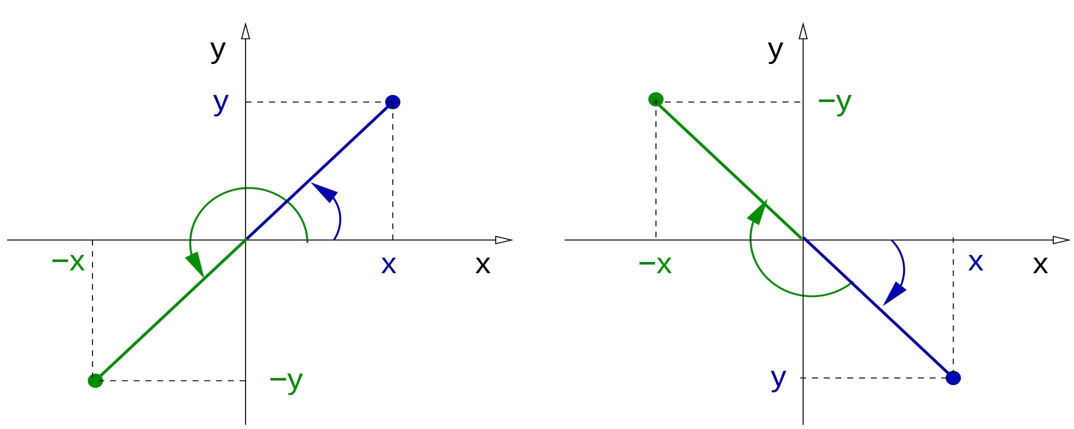
$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right).$$

Proof: $x^2 + y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta) = r^2$; $r \geqslant 0$ implies $r = \sqrt{x^2 + y^2}$. Finally, $x/y = \tan(\theta)$. □

Transformation rules Polar-Cartesian.

Remark:

- If (x, y) satisfies either $x \geqslant 0$, $y \geqslant 0$, or $x \leqslant 0$, $y \leqslant 0$, then $\theta = \arctan(x/y)$ is in the first quadrant.
- If (x, y) satisfies either $x \geqslant 0$, $y \leqslant 0$, or $x \leqslant 0$, $y \geqslant 0$, then $\theta = \arctan(x/y)$ is in the fourth quadrant.



Polar coordinates (Sect. 11.3)

- Review: Arc-length of a curve.
- Polar coordinates definition.
- Transformation rules Polar-Cartesian.
- **Examples of curves in polar coordinates.**

Examples of curves in polar coordinates

Example

Find the equation in polar coordinates of a circle radius 3 at $(0,0)$.

Solution: In Cartesian coordinates the equation is

$$x^2 + y^2 = 3^2, \quad r = \sqrt{x^2+y^2} \quad \Rightarrow \quad \begin{cases} r = 3, \\ \theta \in [0, 2\pi). \end{cases} \quad \triangleleft$$

Example

Find the equation in polar coordinates of the line $y = \sqrt{3}\,x$.

Solution: From the transformation laws,

$$\theta = \arctan(y/x) = \arctan\big(\sqrt{3}\big) \quad \Rightarrow \quad \begin{cases} \theta = \pi/3, \\ r \in \mathbb{R}, \end{cases} \quad \triangleleft$$

Examples of curves in polar coordinates

Example

Find the equation in polar coordinates of the circle $x^2 + (y-3)^2 = 9$.

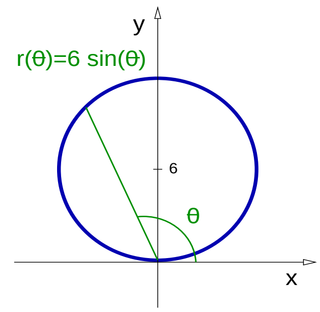
Solution: Expand the square in the equation of the circle,

$$x^2 + y^2 - 6y + 9 = 9 \quad \Rightarrow \quad x^2 + y^2 = 6y.$$

Recall: $x = r\cos(\theta)$, and $y = r\sin(\theta)$, therefore $x^2 + y^2 = r^2$,

$$r^2 = 6r\sin(\theta) \quad \Rightarrow \quad r = 6\sin(\theta),$$

and $\theta \in [0, \pi]$. $\triangleleft$



Examples of curves in polar coordinates

Example

Find the equation of the curve in Cartesian coordinates for $r = 4\cos(\theta)$, for $\theta \in [-\pi/2, \pi/2]$.

Solution: Multiply by r the whole equation, $r^2 = 4r\cos(\theta)$.

Recall: $x = r\cos(\theta)$, and $y = r\sin(\theta)$, therefore $x^2 + y^2 = r^2$,

$$x^2 + y^2 = 4x \quad \Rightarrow \quad x^2 - 4x + y^2 = 0.$$

Complete the square:

$$\left[x^2 - 2\left(\frac{4}{2}\right)x + 4\right] - 4 + y^2 = 0$$

$$(x-2)^2 + y^2 = 4.$$

This is the equation of a circle radius $r = 2$ with center at $(2, 0)$. $\triangleleft$