Arc-length of a curve on the plane (Sect. 11.2)

- Review: Parametric curves on the plane.
- The slope of tangent lines to curves.
- The arc-length of a curve.
- The arc-length function and differential.

Review: Parametric curves on the plane

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

Example

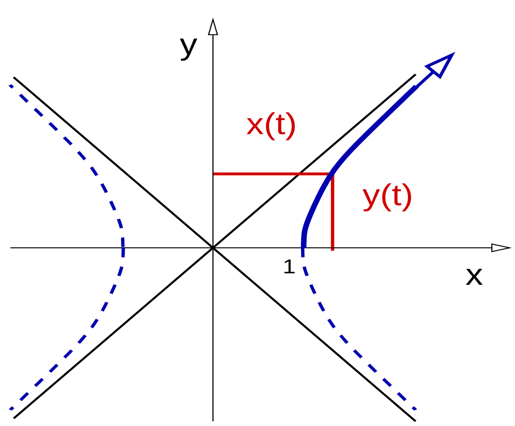
Describe the curve $x(t) = \cosh(t)$, $y(t) = \sinh(t)$, for $t \in [0, \infty)$.

Solution:

$$[x(t)]^2 - [y(t)]^2 =$$

$$\cosh^2(t) - \sinh^2(t) = 1.$$

This is a portion of a hyperbola with asymptotes $y = \pm x$, starting at $(1, 0)$. ◁



Review: Parametric curves on the plane

Definition

A *cycloid* with parameter $a > 0$ is the curve given by

$$x(t) = a(t - \sin(t)), \quad y(t) = a(1 - \cos(t)), \quad t \in \mathbb{R}.$$

Remark: From the equation of the cycloid we see that

$$x(t) - at = a\sin(t), \qquad y(t) - a = a\cos(t).$$

Therefore, $[x(t) - at]^2 + [y(t) - a]^2 = a^2$.

Remarks:

- This is not the equation of a circle.
- The point $(x(t), y(t))$ belongs to a moving circle.
- The cycloid played an important role in designing precise pendulum clocks, needed for navigation in the 17th century.

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The slope of tangent lines to curves

Definition

A curve defined by the parametric function values $(x(t), y(t))$, for $t \in I \subset \mathbb{R}$, is *differentiable* iff each function x and y is differentiable on the interval I.

Theorem

Assume that the curve defined by the graph of the function $y = f(x)$, for $x \in (a, b)$, can be described by the parametric function values $(x(t), y(t))$, for $t \in I \subset \mathbb{R}$. If this parametric curve is differentiable and $x'(t) \neq 0$ for $t \in I$, then holds

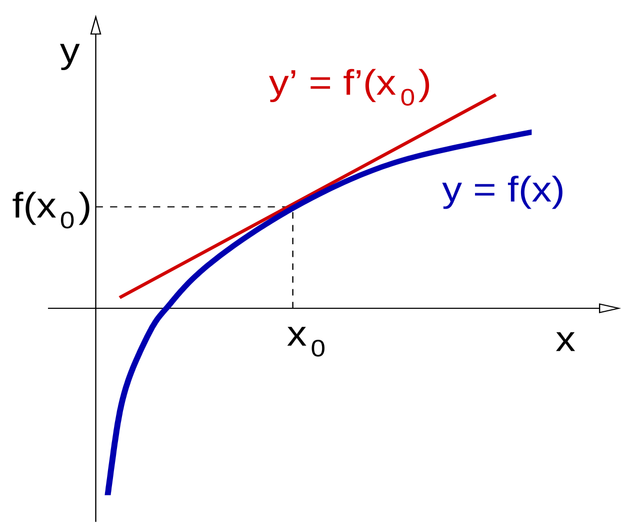
$$\frac{df}{dx} = \frac{(dy/dt)}{(dx/dt)}.$$

Proof: Express $y(t) = f(x(t))$, then

$$\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt} \quad \Rightarrow \quad \frac{df}{dx} = \frac{(dy/dt)}{(dx/dt)}. \qquad \square$$

The slope of tangent lines to curves

Remark: The formula $\frac{df}{dx} = \frac{(dy/dt)}{(dx/dt)}$ provides an alternative way to find the slope of the line tangent to the graph of the function f.



The slope of tangent lines to curves

Example

Find the slope of the tangent lines to a circle radius r at $(0,0)$.

Solution: The equation of the circle is $x^2 + y^2 = r^2$.
One possible set of parametric equations are:

$$x(t) = r\cos(nt), \qquad y(t) = r\sin(nt), \qquad n \geqslant 1.$$

The derivatives of the parametric functions are

$$x'(t) = -nr\sin(nt), \qquad y'(t) = nr\cos(nt).$$

The slope of the tangent lines to the circle at $x_0 = \cos(nt_0)$ is

$$y'(x_0) = \frac{y'(t_0)}{x'(t_0)} = \frac{-nr\cos(nt_0)}{nr\sin(nt_0)} \quad \Rightarrow \quad y'(x_0) = -\frac{1}{\tan(nt_0)}.$$

Remark: In the first quadrant holds $y'(x_0) = \dfrac{-x_0}{\sqrt{1-(x_0)^2}}$. ◁

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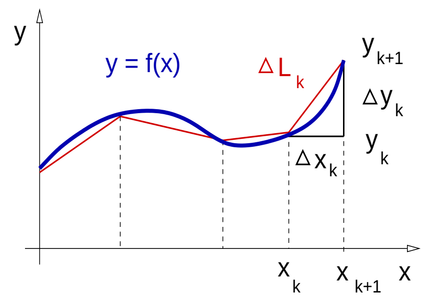
The arc-length of a curve

Definition

The *length* or *arc length* of a curve in the plane or in space is the limit of the polygonal line length, as the polygonal line approximates the original curve.



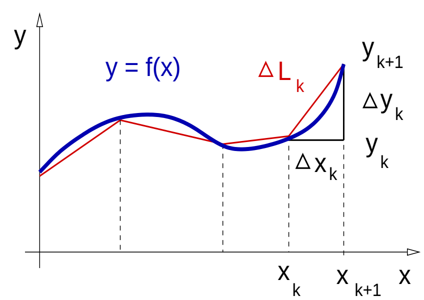
Theorem

The arc-length of a continuously differentiable curve $(x(t), y(y))$, *for* $t \in [a, b]$ *is the number*

$$L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

The arc-length of a curve

Idea of the Proof: The curve length is the limit of the polygonal line length, as the polygonal line approximates the original curve.



$$L_N = \sum_{n=0}^{N-1} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \qquad \{a = t_0, t_1, \cdots, t_{N-1}, t_N = b\},$$

$$L_N \simeq \sum_{n=0}^{N-1} \sqrt{[x'(t_k^*)]^2 + [y'(t_k^*)]^2}\, \Delta t_k,$$

$$L_N \xrightarrow{N\to\infty} L = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

□

The arc-length of a curve

Example

Find the length of the curve $\big(r\cos(t), r\sin(t)\big)$, for $r > 0$ and $t \in [\pi/4, 3\pi/4]$. (Quarter of a circle.)

Solution: Compute the derivatives $\big(-r\sin(t), r\cos(t)\big)$. The length of the curve is given by the formula

$$L = \int_{\pi/4}^{3\pi/4} \sqrt{\big[-r\sin(t)\big]^2 + \big[r\cos(t)\big]^2}\,dt$$

$$L = \int_{\pi/4}^{3\pi/4} \sqrt{r^2\Big(\big[-\sin(t)\big]^2 + \big[\cos(t)\big]^2\Big)}\,dt = \int_{\pi/4}^{3\pi/4} r\,dt.$$

Hence, $L = \dfrac{\pi}{2}\,r$. (The length of quarter circle of radius r.) $\triangleleft$

The arc-length of a curve

Example

Find the length of the spiral $\big(t\cos(t), t\sin(t)\big)$, for $t \in [0, t_0]$.

Solution: The derivative of the parametric curve is

$$\big(x'(t), y'(t)\big) = \Big(\big[-t\sin(t) + \cos(t)\big], \big[t\cos(t) + \sin(t)\big]\Big),$$

$$\begin{aligned}(x')^2 + (y')^2 = {} & \big[t^2\sin^2(t) + \cos^2(t) - 2t\sin(t)\cos(t)\big] \\ & + \big[t^2\cos^2(t) + \sin^2(t) + 2t\sin(t)\cos(t)\big]\end{aligned}$$

We obtain $(x')^2 + (y')^2 = t^2 + 1$. The curve length is given by

$$L(t_0) = \int_0^{t_0} \sqrt{1+t^2}\,dt = \Big[\frac{t}{2}\sqrt{1+t^2} + \frac{1}{2}\ln\big(t + \sqrt{1+t^2}\,\big)\Big]\Big|_0^{t_0}.$$

We conclude that $L(t_0) = \dfrac{t_0}{2}\sqrt{1+t_0^2} + \dfrac{1}{2}\ln\big(t_0 + \sqrt{1+t_0^2}\,\big)$. $\triangleleft$

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The arc-length function and differential

Remark: The previous example suggests to introduce the length function of a curve.

Definition

The *arc-length function* of a continuously differentiable curve given by $\big(x(t), y(t)\big)$ for $t \in [t_0, t_1]$ is given by

$$L(t) = \int_{t_0}^{t} \sqrt{\big[x'(\tau)\big]^2 + \big[y'(\tau)\big]^2}\, d\tau.$$

Remarks:

(a) The value $L(t)$ of the length function is the length along the curve $\big(x(t), y(t)\big)$ from t_0 to t.

(b) If the curve is the position of a moving particle as function of time, then the value $L(t)$ is the distance traveled by the particle from the time t_0 to t.

The arc-length function and differential

Remark: The arc-length differential is the differential of the arc-length function, that is,

$$dL = \sqrt{\big[x'(t)\big]^2 + \big[y'(t)\big]^2}\, dt.$$

This is a useful notation.

Example

Find the length of $x(t) = (2t+1)^{3/2}/3$, $y(t) = t + t^2$ for $t \in [0, 1]$.

Solution: We first compute the length differential,

$$dL = \Big[\frac{1}{3}\,\frac{3}{2}(2t+1)^{1/2}\,2\Big]^2 + \big[1+2t\big]^2 = (2t+1) + 1 + 4t + 4t^2$$

$$L = \int_0^1 (4t^2 + 6t + 2)\, dt = \Big(\frac{4t^3}{3} + 3\,t^2 + 2t\Big)\Big|_0^1 = \frac{19}{3}. \qquad \triangleleft$$