Parametrizations of curves on a plane (Sect. 11.1)

- Review: Curves on the plane.
- Parametric equations of a curve.
- Examples of curves on the plane.
- The cycloid.

Review: Curves on the plane

Remarks:

- Curves on a plane can be described by the set of points (x, y) solutions of an equation

$$F(x, y) = 0.$$

- A particular case is the graph of a function $y = f(x)$.
  In this case: $F(x, y) = y - f(x)$.

Example

- Circle centered at $P = (0, 0)$ radius r:

$$x^2 + y^2 = r^2.$$

- Circle centered at $P = (x_0, y_0)$ radius r:

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

Review: Curves on the plane

Example

- An ellipse centered at $P = (0, 0)$ with radius a and b,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

  A sphere is the particular case $a = b = r$.

- A hyperbola with asymptotes $y = \pm x$,

$$x^2 - y^2 = 1.$$

- A hyperbola with asymptotes $y = \pm \frac{b}{a} x$,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

Review: Curves on the plane

Example

- A parabola with minimum at $(0, 0)$,

$$y = x^2.$$

- A parabola with minimum at (a, b),

$$y = c\,(x - a)^2 + b, \qquad c > 0.$$

- A parabola with maximum at (a, b),

$$y = -c\,(x - a)^2 + b, \qquad c > 0.$$

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- **Parametric equations of a curve.**
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Parametric equations of a curve

Remarks:

- A curve on a plane can always be thought as the motion of a particle as function of time.
- Every curve given by $F(x, y) = 0$ can be described as the set of points $\big(x(t), y(t)\big)$ traveled by a particle for $t \in [a, b]$.

Definition

A curve on the plane is given in *parametric form* iff it is given by the set of points $\big(x(t), y(t)\big)$, where the parameter $t \in I \subset \mathbb{R}$.

Remark: If the interval I is closed, $I = [a, b]$, then $(x(a), y(a))$ and $(x(b), y(b))$ are called the *initial* and *terminal* points of the curve.

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Examples of curves on the plane

Example

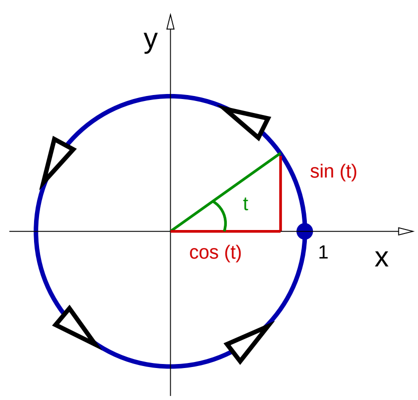
Describe the curve $x(t) = \cos(t), \;\; y(t) = \sin(t)$, for $t \in [0, 2\pi]$.

Solution:

The functions x and y above satisfy the equation

$$[x(t)]^2 + [y(t)]^2 =$$

$$\cos^2(t) + \sin^2(t) = 1.$$



This is a circle.

This is the equation of a circle radius $r = 1$, centered at $(0, 0)$. The circle is traversed in counterclockwise direction, starting and ending at $(1, 0)$. ◁

Examples of curves on the plane

Example

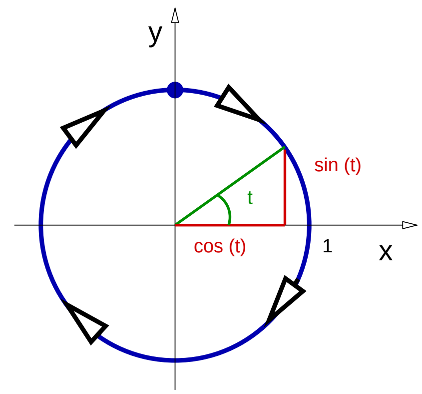
Describe the curve $x(t) = \sin(t)$, $y(t) = \cos(t)$, for $t \in [0, 2\pi]$.

Solution:

The functions x and y above satisfy the equation

$$[x(t)]^2 + [y(t)]^2 =$$

$$\sin^2(t) + \cos^2(t) = 1.$$



This is a circle.

This is the equation of a circle radius $r = 1$, centered at $(0, 0)$. The circle is traversed in clockwise direction, starting and ending at $(0, 1)$. $\triangleleft$

Examples of curves on the plane

Example

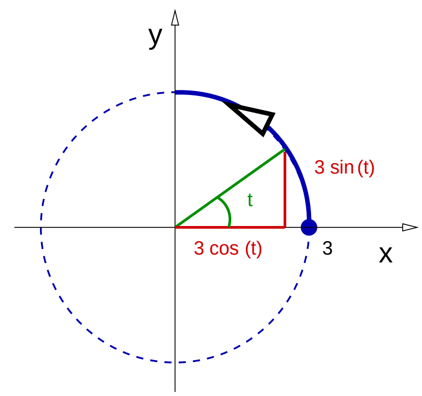
Describe the curve $x(t) = 3\cos(t)$, $y(t) = 3\sin(t)$, for $t \in [0, \pi/2]$.

Solution:

The functions x and y above satisfy the equation

$$[x(t)]^2 + [y(t)]^2 =$$

$$3^2 \cos^2(t) + 3^2 \sin^2(t) = 3^2.$$



This is a portion of a circle.

This is the equation of a $1/4$ circle radius $r = 3$, centered at $(0, 0)$. The circle is traversed in counterclockwise direction, starting at $(3, 0)$ and ending at $(0, 3)$. $\triangleleft$

Examples of curves on the plane

Example

Describe the curve $x(t) = 3\cos(2t)$, $y(t) = 3\sin(2t)$, for $t \in [0, \pi/2]$.

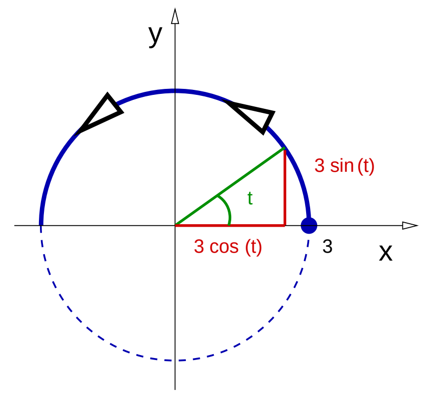
Solution:

The functions x and y above satisfy the equation

$$[x(t)]^2 + [y(t)]^2 =$$

$$3^2\cos^2(2t) + 3^2\sin^2(2t) = 3^2.$$

This is a portion of a circle.



This is the equation of a $1/2$ circle radius $r = 3$, centered at $(0, 0)$. The circle is traversed in counterclockwise direction, starting at $(3, 0)$ and ending at $(-3, 0)$. $\lhd$

Examples of curves on the plane

Example

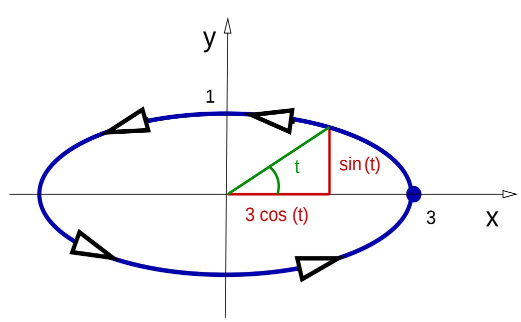
Describe the curve $x(t) = 3\cos(t)$, $y(t) = \sin(t)$, for $t \in [0, 2\pi]$.

Solution:

The functions x and y above satisfy the equation

$$\frac{[x(t)]^2}{3^2} + [y(t)]^2 =$$

$$\cos^2(t) + \sin^2(t) = 1.$$



This is an ellipse.

This is the equation of an ellipse with x-radius 3 and y-radius 1, centered at $(0, 0)$. The ellipse is traversed in counterclockwise direction, starting and ending at $(3, 0)$. $\lhd$

Examples of curves on the plane

Example

Describe the curve $x(t) = \cosh(t)$, $y(t) = \sinh(t)$, for $t \in [0, \infty)$.

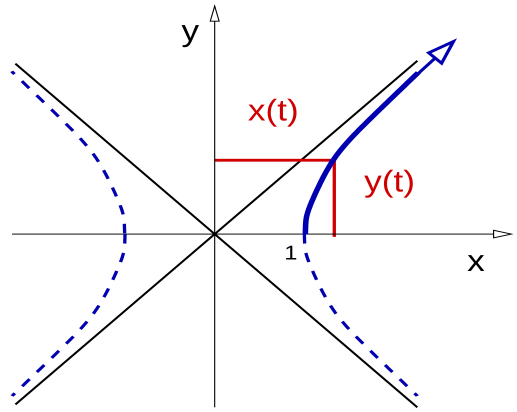
Solution:

The functions x and y above satisfy the equation

$$[x(t)]^2 - [y(t)]^2 =$$

$$\cosh^2(t) - \sinh^2(t) = 1.$$

This is a portion of a hyperbola.



This is the equation of a hyperbola with asymptotes $y = \pm x$.
The hyperbola portion starts at $(1, 0)$. $\triangleleft$

Examples of curves on the plane

Example

Describe the curve $x(t) = -\sec(t)$, $y(t) = \tan(t)$, for $t \in [0, \pi/2)$.

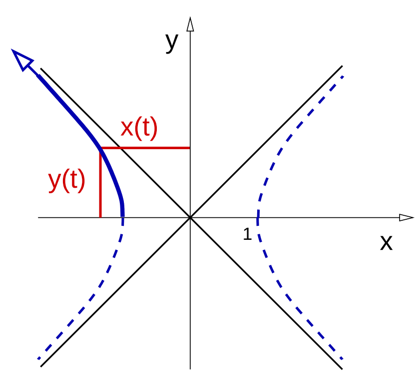
Solution:

Recall: $\tan^2(t) + 1 = \sec^2(t)$.
Therefore,

$$[x(t)]^2 - [y(t)]^2 =$$

$$\sec^2(t) - \tan^2(t) = 1.$$

This is a portion of a hyperbola.



This is the equation of a hyperbola with asymptotes $y = \pm x$.
The hyperbola portion starts at $(-1, 0)$. $\triangleleft$

Examples of curves on the plane

Example

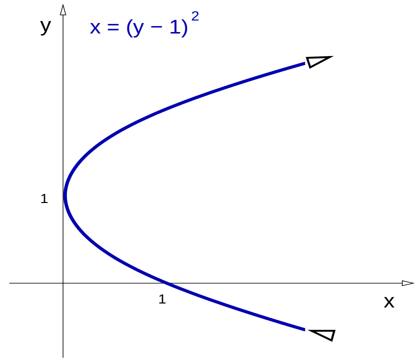
Describe the curve $x(t) = t^2$, $y(t) = t + 1$, for $t \in (-\infty, \infty)$.

Solution:

Since $t = y - 1$, then

$$x = (y - 1)^2.$$

This is a parabola.



This is the equation of a parabola opening to the right. Passing through $(1, 0)$ (for $t = -1$), then $(0, 1)$ (for $t = 0$), and then $(1, 2)$ (for $t = 1$). $\triangleleft$

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The cycloid

Definition

A *cycloid* with parameter $a > 0$ is the curve given by

$$x(t) = a(t - \sin(t)), \quad y(t) = a(1 - \cos(t)), \quad t \in \mathbb{R}.$$

Remark: From the equation of the cycloid we see that

$$x(t) - at = a\sin(t), \qquad y(t) - a = a\cos(t).$$

Therefore, $[x(t) - at]^2 + [y(t) - a]^2 = a^2$.

Remarks:

- This is not the equation of a circle.
- The point $(x(t), y(t))$ belongs to a moving circle.
- The cycloid played an important role in designing precise pendulum clocks, needed for navigation in the 17th century.