Binomial functions and Taylor series (Sect. 10.10)

- Review: The Taylor Theorem.
- The binomial function.
- Evaluating non-elementary integrals.
- The Euler identity.
- Taylor series table.

Review: The Taylor Theorem

Recall: If $f : D \to \mathbb{R}$ is infinitely differentiable, and a, $x \in D$, then

$$f(x) = T_n(x) + R_n(x),$$

where the *Taylor polynomial* T_n and the *Remainder function* R_n are

$$T_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

$$R_n(x) = \frac{f^{(n+1)}(c(x))}{(n+1)!}(x-a)^{n+1}, \quad \text{with} \quad c \in (a, x).$$

Furthermore, if $R_n(x) \to 0$ as $n \to \infty$ for every $x \in I \subset D$, then the *Taylor series* centered at $x = a$, $T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, converges to the function f on the interval I, and $f(x) = T(x)$.

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The binomial function

Definition

The *binomial function* is a function of the form

$$f_m(x) = (1+x)^m, \qquad m \in \mathbb{R}.$$

Example

Find the Taylor polynomial T_3 centered at $a = 0$ of f_m.

Solution: The derivatives of the function $f(x) = (1+x)^m$ are

$$f'(x) = m(1+x)^{(m-1)}, \quad f''(x) = m(m-1)(1+x)^{(m-2)},$$

$$f^{(3)}(x) = m(m-1)(m-2)\,(1+x)^{(m-3)}.$$

$$T_3(x) = 1 + m\,x + \frac{m(m-1)}{2!}\,x^2 + \frac{m(m-1)(m-2)}{3!}\,x^3. \quad \triangleleft$$

The binomial function

Remark: If m is a positive integer, then the binomial function f_m is a polynomial, therefore the Taylor series is the same polynomial, hence the Taylor series has only the first $m+1$ terms non-zero.

Example

Find the Taylor series of $f_2(x) = (1+x)^2$.

Solution: Expanding the the binomial $f_2(x) = (1+x)^2$,

$$f_2(x) = 1 + 2x + x^2 \quad \Rightarrow \quad f'(x) = 2 + 2x, \quad f''(x) = 2.$$

Since all derivatives higher or equal the third vanish,

$$T(x) = 1 + f'(0)\,x + \frac{f''(0)}{2}\,x^2 \quad \Rightarrow \quad T(x) = 1 + 2x + x^2.$$

That is, $f_2(x) = T(x)$. $\triangleleft$

The binomial function

Remark: If m is not a positive integer, then the Taylor series of the binomial function has infinitely many non-zero terms.

Theorem

The Taylor series for the binomial function $f_m(x) = (1+x)^m$, with m not a positive integer converges for $|x| < 1$ and is given by

$$T(x) = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n,$$

with the binomial coefficients $\binom{m}{1} = m$, $\binom{m}{2} = \frac{m(m-1)}{2!}$, and

$$\binom{m}{n} = \frac{m(m-1)\cdots(m-(n-1))}{n!}.$$

The binomial function

Proof: The n-derivative of the binomial function is

$$f^{(n)}(x) = m(m-1)\cdots(m-(n-1))\,(1+x)^{(m-n)},$$

therefore, the n-Taylor coefficient at $a = 0$ is

$$\frac{f^{(n)}(0)}{n!} = \frac{m(m-1)\cdots(m-(n-1))}{n!} = \binom{m}{n}.$$

Since $f(0) = 1$, the Taylor series of the binomial function is

$$T(x) = 1 + \sum_{n=1}^{\infty} \binom{m}{n} x^n,$$

The ratio test: $\dfrac{\left|x^{n+1}\binom{m}{n+1}\right|}{\left|x^n\binom{m}{n}\right|} = \left|x\dfrac{m-n}{(n+1)}\right| \to |x|$ as $n \to \infty$.

Therefore, the series converges for $|x| < 1$. □

The binomial function

Example

Find the Taylor series of the binomial function $f(x) = (1+x)^{1/2}$.

Solution: Compute the binomial coefficients $\binom{1/2}{n}$: $\binom{1/2}{1} = \dfrac{1}{2}$,

$$\binom{1/2}{2} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = \frac{\left(-\frac{1}{4}\right)}{2} = -\frac{1}{8},$$

$$\binom{1/2}{3} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} = \frac{\left(\frac{3}{8}\right)}{6} = \frac{1}{16}.$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots.$$

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The binomial function

Example

Find the Taylor series of the binomial function $f(x) = (1 - x)^{1/2}$.

Solution: Substitute x by $-x$ in $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$.

We obtain: $\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \cdots$. $\triangleleft$

Example

Find the Taylor series of the binomial function $f(x) = (1 - x^2)^{1/2}$.

Solution: Substitute x by $-x^2$ in $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \cdots$.

We obtain: $\sqrt{1-x^2} = 1 - \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^6}{16} - \cdots$. $\triangleleft$

The binomial function

Example

Find the Taylor series of the binomial function $f(x) = (1 + x)^{1/3}$.

Solution: Compute the binomial coefficients $\binom{1/3}{n}$: $\binom{1/3}{1} = \frac{1}{3}$,

$$\binom{1/3}{2} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3} - 1\right)}{2!} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} = \frac{\left(-\frac{2}{9}\right)}{2} = -\frac{1}{9},$$

$$\binom{1/3}{3} = \frac{\left(\frac{1}{3}\right)\left(\frac{1}{3} - 1\right)\left(\frac{1}{3} - 2\right)}{3!} = \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} = \frac{\left(\frac{10}{27}\right)}{6} = \frac{5}{81}.$$

$$\sqrt[3]{1+x} = 1 + \frac{x}{3} - \frac{x^2}{9} + \frac{5}{81}x^3 - \cdots.$$ $\triangleleft$

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Evaluating non-elementary integrals

Remark: Non-elementary integrals can be evaluated integrating term by term the integrand Taylor series.

Example

Approximate the integral $I = \int_0^1 e^{-x^2}\,dx$.

Solution: Recall the Taylor series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$.

Substitute x by $-x^2$ in the Taylor series,

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots .$$

$$\int e^{-x^2}\,dx = x - \frac{x^3}{3} + \frac{x^5}{(2!)(5)} - \frac{x^7}{(3!)(7)} + \cdots .$$

$$\int_0^1 e^{-x^2}\,dx = 1 - \frac{1}{3} + \frac{1}{(2!)(5)} - \frac{1}{(3!)(7)} + \cdots . \qquad \triangleleft$$

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The Euler identity

Remark: The Taylor expansions

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots, \quad \sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$

imply that

$$\cos(\theta) + i\sin(\theta) = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \cdots,$$

$$\cos(\theta) + i\sin(\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots,$$

This and $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ suggest the definition:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

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Taylor series table

Remark: Table of frequently used Taylor series.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots, \qquad |x| < 1,$$

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots, \qquad |x| < 1,$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \qquad |x| < \infty,$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots, \qquad |x| < \infty,$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \qquad |x| < \infty.$$