Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- The Taylor Theorem.
- Using the Taylor series.
- Estimating the remainder.

Review: Taylor series and polynomials

Definition

The *Taylor series* and *Taylor polynomial* order n centered at $a \in D$ of a differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ are given by

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k, \quad T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (x-a)^k.$$

Remarks:

- The Taylor series may or may not converge.
- $T_1(x) = f(a) + f'(a)(x-a)$ is the linearization of f.
- The Taylor polynomial is called of order n instead of degree n, because $f^{(n)}(a)$ may vanish.
- The particular case $a = 0$ is called the *Maclaurin series* and the $n+1$ *Maclaurin polynomial*, respectively.

Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- **The Taylor Theorem.**
- Using the Taylor series.
- Estimating the remainder.

The Taylor Theorem

Remark: The Taylor polynomial and Taylor series are obtained from a generalization of the Mean Value Theorem: If $f : [a, b] \to \mathbb{R}$ is differentiable, then there exits $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{(b - a)} = f'(c) \quad \Leftrightarrow \quad f(b) = f(a) + f'(c)\,(b - a).$$

Theorem (Taylor's Theorem)

If $f : [a, b] \to \mathbb{R}$ is $(n + 1)$-times continuously differentiable, then there there exists $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)\,(b - a) + \frac{f''(a)}{2}\,(b - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}\,(b - a)^n + \frac{f^{(n+1)}(c)}{(n + 1)!}\,(b - a)^{n+1}.$$

The Taylor Theorem

Remark: The Taylor Theorem is usually applied for a fixed point a, while the point $b = x$ is used as an independent variable:

$$f(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the *remainder function* R_n is given by

$$R_n(x) = \frac{f^{(n+1)}(c(x))}{(n+1)!}(x-a)^{n+1}, \quad \text{with} \quad c \in (a, x).$$

Remark: The point $c \in (a, x)$ also depends on x.

Remark: We can use the Taylor polynomial to write that

$$f(x) = T_n(x) + R_n(x).$$

The Taylor Theorem

Corollary

Let $f : D \to \mathbb{R}$ be infinitely differentiable with Taylor polynomials T_n and remainders R_n, that is, for $n \geqslant 1$ holds

$$f(x) = T_n(x) + R_n(x).$$

If $R_n(x) \to 0$ as $n \to \infty$ for $x \in D$, then the Taylor series centered at $x = a$ converges on D to the function values $f(x)$, that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Remark: Without knowing $c(x)$ it is often possible to estimate

$$R_n(x) = \frac{f^{(n+1)}(c(x))}{(n+1)!}(x-a)^{n+1}.$$

Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- The Taylor Theorem.
- **Using the Taylor series.**
- Estimating the remainder.

Using the Taylor series

Example

Show that the Taylor series of $f(x) = e^x$ centered at $a = 0$ converges on $\mathbb{R}$.

Solution: Since $f^{(n)}(x) = e^x$ and $a = 0$, then $f^{(n)}(0) = 1$, and

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x),$$

where $R_n(x) = e^{c(x)} \dfrac{x^{n+1}}{(n+1)!}$. Since $f(x)$ is increasing,

$$|R_n(x)| = e^{c(x)} \frac{|x|^{n+1}}{(n+1)!} \leqslant e^{|x|} \frac{|x|^{n+1}}{(n+1)!} \to 0, \quad \text{as} \quad n \to \infty.$$

for every $x \in \mathbb{R}$. Then $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots.$ $\lhd$

Using the Taylor series

Example

Find the Taylor series of $f(x) = e^{x^2}$ centered at $a = 0$.

Solution: We use the Taylor series $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$ for $y = x^2$,

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \cdots . \qquad \triangleleft$$

Example

Find the Taylor series of $f(x) = e^{-x^2}$ centered at $a = 0$.

Solution: We substitute x^2 by $-(x^2)$ in the example above,

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots . \qquad \triangleleft$$

Using the Taylor series

Example

Find the Taylor series of $f(x) = \frac{3x^2}{(1-x)^3}$ at $a = 0$ on $(-1, 1)$.

Solution: The straightforward way is to compute the derivatives $f^{(n)}(x)$. A simpler ways is to realize that

$$f(x) = 3x^2 \left[\frac{1}{(1-x)^3}\right] = 3x^2 \, \frac{1}{2} \left[\frac{1}{(1-x)^2}\right]' = \frac{3}{2} \, x^2 \left[\frac{1}{(1-x)}\right]''$$

Recall that for $|x| < 1$ holds $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Hence,

$$f(x) = \frac{3}{2} \, x^2 \Big[\sum_{n=0}^{\infty} x^n\Big]'' = \frac{3}{2} \, x^2 \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{3}{2} \sum_{n=2}^{\infty} n(n-1) x^n.$$

We conclude: $f(x) = \frac{3}{2}\big[2x^2 + (3)(2)x^3 + (4)(3)x^4 + \cdots\big]$. $\triangleleft$

Using the Taylor series

Example

Find the Taylor series of $f(x) = \cos(2\sqrt{x})$ at $a = 0$ on $(-1, 1)$.

Solution: If $y(x) = \cos(x)$, $y'(x) = -\sin(x)$, $y''(x) = -\cos(x)$, $y'''(x) = \sin(x)$, $y^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1, \quad y'''(0) = 0, \quad y^{(4)}(0) = 1.$$

Then, $\cos(x) = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

Substitute x by $2\sqrt{x}$ above, $\cos\big(2\sqrt{x}\big) = \displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{(2\sqrt{x})^{2n}}{(2n)!}$.

$$\cos\big(2\sqrt{x}\big) = \sum_{n=0}^{\infty} \frac{(-4x)^n}{(2n)!} = 1 - \frac{4x}{2!} + \frac{(4x)^2}{4!} - \frac{(4x)^3}{6!} + \cdots . \quad \triangleleft$$

Convergence of Taylor Series (Sect. 10.9)

- Review: Taylor series and polynomials.
- The Taylor Theorem.
- Using the Taylor series.
- **Estimating the remainder.**

Estimating the remainder

Theorem

Let $f : D \to \mathbb{R}$ be infinitely differentiable with Taylor polynomials T_n and remainders R_n centered at $a \in D$, that is, for $n \geqslant 1$ holds

$$f(x) = T_n(x) + R_n(x).$$

If $|f^{(n+1)}(y)| \leqslant M$ for all y such that $|y - a| \leqslant |x - a|$, then

$$|R_n(x)| \leqslant M \frac{|x - a|^{n+1}}{(n+1)!}.$$

Furthermore, if the inequality above holds for every $n \geqslant 1$, then the Taylor series $T(x)$ converges to $f(x)$.

Estimating the remainder

Example

Estimate the maximum error made in approximating $f(x) = e^x$ by $T_2(x) = 1 + x + \dfrac{x^2}{2}$ over the interval $[-2, 2]$.

Solution: We use the formula $|R_n(x)| \leqslant M \dfrac{|x - a|^{n+1}}{(n+1)!}$, for $a = 0$, where M satisfies $|f^{(n+1)}(y)| \leqslant M$, for $|y| \leqslant |2 - 0| = 2$.

Since $f(x) = e^x$, and $n = 2$, and $f^{(3)}(x) = e^x$, then

$$|f^{(3)}(y)| \leqslant e^2 = M \quad \text{for} \quad |y - 0| \leqslant |2 - 0| = 2.$$

Therefore, the smallest bound for R_2 in $[-2, 2]$ is

$$|R_2(x)| \leqslant e^2 \frac{|x|^3}{3!} \leqslant e^2 \frac{2}{6} \quad \Rightarrow \quad |R_2(x)| \leqslant \frac{e^2}{3}. \qquad \triangleleft$$