Taylor Series (Sect. 10.8)

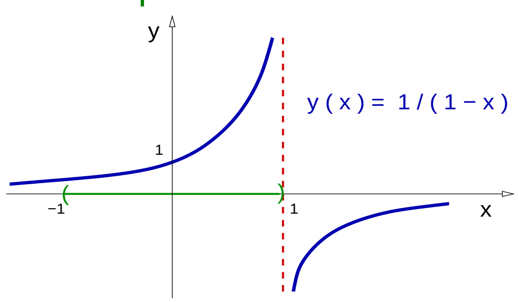
- Review: Power series define functions.
- Functions define power series.
- Taylor series of a function.
- Taylor polynomials of a function.

Review: Power series define functions

Remarks:

- Power series define functions on domains where the series converge.
- Given a sequence $\{c_n\}$ and a number x_0, the function $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ is defined for every $x \in \mathbb{R}$ where the series converges.

Example



The power series $y(x) = \sum_{n=0}^{\infty} x^n$ converges to the function $f(x) = \dfrac{1}{1-x}$ only on the domain given by $|x| < 1$. $\triangleleft$

Review: Power series define functions

Theorem (Term by term derivation and integration)

If the power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ *has radius of convergence* $\rho > 0$, *then function* y *is both differentiable with derivative*

$$y'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{(n-1)},$$

and function y *is integrable with primitive*

$$\int y(x)\, dx = \sum_{n=0}^{\infty} \frac{(x - x_0)^{(n+1)}}{(n+1)} + c,$$

where both expressions above converge on $(x_0 - \rho, x_0 + \rho)$.

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Functions define power series

Theorem

If an infinitely differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ has a power series representation at $a \in D$ with convergence radius $\rho > 0$,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

then the series coefficients are given by

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Remark: If we only assume that f is infinitely differentiable, we can always construct the series

$$y(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

However, it is not clear whether this series converges at all, and if it does, whether it satisfies that $f(x) = y(x)$ for $x \neq a$.

Functions define power series

Remark: The proof is simple because the assumptions are big.

Proof: Since the function f has a power series representation,

$$f(x) = c_0 + c_1 (x-a) + c_2 (x-a)^2 + c_3 (x-a)^3 + \cdots$$

we can differentiate on both sides many times,

$$f'(x) = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + 4c_4 (x-a)^3 + \cdots$$
$$f''(x) = 2c_2 + (3)(2)c_3 (x-a) + (4)(3) (x-a)^2 + \cdots$$

And evaluating the expressions above at $x = a$ we have

$$f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2, \quad f^{(n)}(a) = n!\, c_n.$$

Therefore, $c_n = \dfrac{f^{(n)}(a)}{n!}$. $\square$

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Taylor series of a function

Remark: The Theorem above suggests the following definition.

Definition

The *Taylor series* centered at $a \in D$ of an infinitely differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is given by

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.$$

Remarks:

- Right now we have no idea whether the Taylor series of a function has a positive radius of convergence.
- And even if the Taylor series has a positive radius of convergence, we do not know if the series T converges to f.
- The particular case $a = 0$ is called the *Maclaurin series*.

Taylor series of a function

Example

Find the Taylor series of the function $f(x) = \dfrac{1}{x}$ centered at $x = 3$.

Solution: We need to compute the function derivatives,

$$f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{(2)(3)}{x^4},$$

therefore: $f^{(n)}(x) = (-1)^n \dfrac{n!}{x^{n+1}}$. Evaluating at $x = 3$,

$$f(3) = \frac{1}{3}, \quad f'(3) = -\frac{1}{3^2}, \quad f''(3) = \frac{2}{3^3}, \quad f^{(n)}(3) = (-1)^n \frac{n!}{3^{n+1}},$$

$$\frac{f^{(n)}(3)}{n!} = \frac{(-1)^n}{3^{n+1}}, \quad T(x) = \frac{1}{3} - \frac{1}{3^2}(x-3) + \frac{1}{3^3}(x-3)^2 - \cdots.$$

We conclude: $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} (x-3)^n.$ ◁

Taylor series of a function

Example

Find the radius of convergence ρ of the Taylor series T centered at $x = a$ of the function $f(x) = \dfrac{1}{x}$.

Solution: It is simple to see that $T(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (x-a)^n.$

The ratio test on $b_n = \left| \dfrac{(-1)^n}{a^{n+1}} (x-a)^n \right|$ says that,

$$\frac{b_{n+1}}{b_n} = \frac{|x-a|^{n+1}}{a^{n+2}} \frac{a^{n+1}}{|x-a|^n} = \frac{|x-a|}{a} \to \frac{|x-a|}{a}.$$

The condition $\displaystyle\lim_{n\to\infty} \frac{b_{n+1}}{b_n} < 1$ implies $\dfrac{|x-a|}{a} < 1$,

that is $|x-a| < a$. We conclude that $\rho = a$. ◁

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Taylor polynomials of a function

Remark: A truncated Taylor series is called a Taylor polynomial.

Definition

The *Taylor polynomial* of order n centered at $a \in D$ of an n-differentiable function $f : D \subset \mathbb{R} \to \mathbb{R}$ is given by

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k.$$

Remarks:

- $T_1(x) = f(a) + f'(a)\,(x-a)$ is the linearization of f.
- The Taylor polynomial is called of order n instead of degree n, because $f^{(n)}(a)$ may vanish.
- The Taylor polynomial of order n centered at $a = 0$ is called the $n+1$ *Maclaurin polynomial*.

Taylor polynomials of a function

Example

Find the first four Maclaurin polynomials of the function $f(x) = e^{3x}$.

Solution: Since $f^{(n)}(x) = 3^n\, e^{3x}$, and the polynomials are centered at $a = 0$, the first 4 Maclaurin polynomials are

$$T_0, \quad T_1, \quad T_2, \quad T_3.$$

Since $T_3(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3$, then

$$T_0(x) = 1, \quad T_1(x) = 1 + 3x, \quad T_2(x) = 1 + 3x + \frac{3^2}{2}\,x^2,$$

$$T_3(x) = 1 + 3x + \frac{3^2}{2}\,x^2 + \frac{3^3}{6}\,x^3. \qquad \lhd$$

Taylor polynomials of a function

Example

Find the first seven Maclaurin polynomials of $f(x) = \cos(x)$.

Solution: $f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f^{(4)}(x) = \cos(x)$, and then the derivatives repeat,

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f^{(4)}(0) = 1.$$

Since $T_n(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f(n)(0)}{n!}\,x^n$, then

$$T_0(x) = 1 = T_1(x), \quad T_2(x) = 1 - \frac{x^2}{2} = T_3(x),$$

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} = T_5(x), \quad T_6(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}. \qquad \lhd$$

Taylor polynomials of a function

Remark: The Taylor polynomial order n centered at any point $x = a$ of a polynomial degree n, say P_n, is the same polynomial. That is, $T_n(x) = P_n(x)$.

Example

Find the T_2 centered at $x = a$ of $f(x) = x^2 + x + 1$.

Solution: Since $f'(x) = 2x + 1$ and $f''(x) = 2$, then

$$f(a) = a^2 + a + 1, \quad f'(a) = 2a + 1, \quad f''(a) = 2.$$

Since $T_2(x) = f(a) + f'(a)\,(x - a) + \frac{f''(a)}{2!}\,(x - a)^2$, then

$$T_2(x) = (a^2 + a + 1) + (2a + 1)\,(x - a) + \frac{2}{2}\,(x - a)^2$$

$$T_2(x) = (a^2 + a + 1) + (2ax - 2a^2 + x - a) + (x^2 - 2ax + a^2).$$

Hence $T_2(x) = 1 + x + x^2$. ◁