Power series (Sect. 10.7)

- Power series definition and examples.
- The radius of convergence.
- The ratio test for power series.
- Term by term derivation and integration.

Power series definition and examples

Definition

A power series centered at x_0 is the function $y : D \subset \mathbb{R} \to \mathbb{R}$

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n, \qquad c_n \in \mathbb{R}.$$

Remarks:

- An equivalent expression for the power series is

$$y(x) = c_0 + c_1 (x - x_0) + c_2 (x - x_0)^2 + c_3 (x - x_0)^3 + \cdots .$$

- A power series centered at $x_0 = 0$ is $y(x) = \sum_{n=0}^{\infty} c_n x^n$, that is,

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

- The domain $D = \{x \in \mathbb{R} \ : \ y(x) \text{ converges.}\}$

Power series definition and examples

Example

The simplest example is $x_0 = 0$, $c_n = 1$, that is

$$y(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots .$$

For every $x \in \mathbb{R}$ this is a geometric series.

Geometric series converge iff $|x| < 1$. and in that case:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \qquad |x| < 1.$$

We conclude that for $|x| < 1$ holds

$$y(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 \cdots \;\Rightarrow\; \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n. \quad \triangleleft$$

Power series definition and examples

Remark:

Another examples of power series $y(x) = \sum_{n=0}^{\infty} c_n \,(x - x_0)^n$.

Example

- $x_0 = 0, \;\; c_n = \frac{1}{n!}$, that is, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots$.
- $x_0 = 1, \;\; \sum_{n=0}^{\infty} \frac{(x-1)^n}{n!} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots$.
- $x = 0, \; c_n = \frac{(-1)^n}{(2n+1)!}$, that is, $y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, x^{(2n+1)}$,

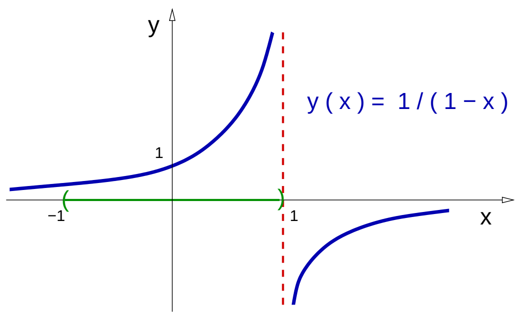
$$y(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Power series definition and examples

Remark: The power series of a function may not be defined on the whole domain of the function.

Example

The function $y(x) = \dfrac{1}{1-x}$ is defined for $x \in \mathbb{R} - \{1\}$.



The power series

$$y(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

converges only for $|x| < 1$. $\triangleleft$

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The radius of convergence.

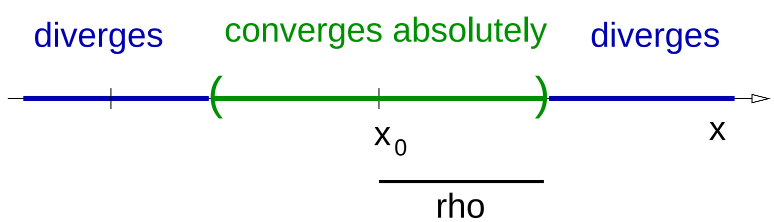
Definition

The power series $y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has *radius of convergence* $\rho \geqslant 0$ iff the following conditions hold:

(a) The series converges absolutely for $|x - x_0| < \rho$;

(b) The series diverges for $|x - x_0| > \rho$.

The *interval of convergence* is the open interval $(x_0 - \rho, x_0 + \rho)$ together with the extreme points $x_0 - \rho$ and $x_0 + \rho$ where the series converges.



The ratio test for power series

Example

Determine the radius of convergence and the interval of convergence of the power series $y(x) = \sum_{n=0}^{\infty} x^n$.

Solution: The power series $y(x)$ is a geometric series for $x \in \mathbb{R}$.

Geometric series converge for $|x| < 1$, and diverge for $|x| > 1$.

Hence the radius of convergence is $\rho = 1$.

For the interval of convergence we need to study $y(1)$ and $y(-1)$.

$$y(1) = 1 + 1 + 1 + 1 + \cdots, \quad y(-1) = 1 - 1 + 1 - 1 + 1 - \cdots$$

Both series diverge, since their partial sums do not converge.

Then the interval of convergence is $I = (-1, 1)$. $\triangleleft$

The ratio test for power series

Example

Determine the radius of convergence of $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution: We fix $x \in \mathbb{R}$ and we use the ratio test on the infinite series $\sum_{n=0}^{\infty} \left|\frac{x^n}{n!}\right|$. Denoting $a_n = \left|\frac{x^n}{n!}\right|$, then

$$\frac{a_{n+1}}{a_n} = \left|\frac{x^{n+1}}{(n+1)!}\right| \left|\frac{n!}{x^n}\right| = \frac{|x^n|\,|x|}{|x^n|} \frac{n!}{(n+1)!} = \frac{|x|}{(n+1)} \to 0$$

as $n \to \infty$, for all $x \in \mathbb{R}$. The radius of convergence $\rho = \infty$. $\lhd$

Remark: The interval of convergence is $I = \mathbb{R}$.

The ratio test for power series

Example

Determine the radius of convergence of $y(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$.

Solution: Fix $x \in \mathbb{R}$ and use the ratio test on the series $\sum_{n=1}^{\infty} \left|\frac{x^n}{n}\right|$.

Denoting $a_n = \left|\frac{x^n}{n}\right|$, then

$$\frac{a_{n+1}}{a_n} = \left|\frac{x^{n+1}}{(n+1)}\right| \left|\frac{n}{x^n}\right| = \frac{|x^n|\,|x|}{|x^n|} \frac{n}{(n+1)} = |x| \frac{n}{(n+1)} \to |x|$$

as $n \to \infty$, for all $x \in \mathbb{R}$. The ratio test says that the series with coefficients $a_n = \left|\frac{x^n}{n}\right|$ converges iff $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.

This is a condition on x, since $|x| = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. So $\rho = 1$. $\lhd$

The ratio test for power series

Example

Determine the interval of convergence of $y(x) = \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n}$.

Solution: Recall: The radius of convergence is $\rho = 1$

This means that for $x \in (-1, 1)$ the power series converges, and for $x \in (-\infty, -1) \cup (1, \infty)$ the series diverges.

We need to study the series for $x = \pm 1$.

$$x = 1 \;\Rightarrow\; y(1) = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n} = -\Big(1 - \frac{1}{2} + \frac{1}{3} - \cdots\Big), \text{ converges.}$$

$$x = -1 \;\Rightarrow\; y(-1) = \sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots\,, \text{ diverges.}$$

The interval of convergence is $I = (-1, 1]$. $\triangleleft$

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The ratio test for power series

Theorem (Ratio test for power series)

Given the power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, *introduce the number* $L = \lim_{n\to\infty} \frac{|c_{n+1}|}{|c_n|}$. *Then, the following statements hold:*

(1) *The power series converges in the domain* $|x - x_0|L < 1$.

(2) *The power series diverges in the domain* $|x - x_0|L > 1$.

(3) *The power series may or may not converge at* $|x - x_0|L = 1$.

Therefore, if $L \neq 0$, *then* $\rho = \frac{1}{L}$ *is the series radius of convergence; if* $L = 0$, *then the radius of convergence is* $\rho = \infty$.

Proof: $\left|\frac{c_{n+1}(x-x_0)^{n+1}}{c_n(x-x_0)^n}\right| = |x - x_0| \frac{|c_{n+1}|}{|c_n|} \to |x - x_0|L.$ □

The ratio test for power series

Example

Determine the radius of convergence of $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{8^n}$.

Solution: Use the ratio test on the series $\sum_{n=0}^{\infty} a_n$ with $a_n = \left|\frac{x^n}{8^n}\right|$.

$$\frac{a_{n+1}}{a_n} = \left|\frac{x^{n+1}}{8^{n+1}}\right|\left|\frac{8^n}{x^n}\right| = \frac{|x^n|\,|x|}{|x^n|}\,\frac{8^n}{8^n\,8} = |x|\,\frac{1}{8} \to \frac{|x|}{8} \quad \text{as } n \to \infty.$$

The ratio test says that the series with coefficients $a_n = \left|\frac{x^n}{n}\right|$ converges if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$, and diverges if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$,

These are a conditions on x, since $\frac{|x|}{8} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$.

The series converges for $|x| < 8$ and diverges for $|x| > 8$.

The radius of convergence is $\rho = 8$. ◁

The ratio test for power series

Example

Determine the radius of convergence of $y(x) = \sum_{n=0}^{\infty} \frac{x^n}{8^n}$.

Solution: Recall: The radius of convergence is $\rho = 8$

This means that for $x \in (-8, 8)$ the power series converges, and for $x \in (-\infty, -8) \cup (8, \infty)$ the series diverges.

We need to study the series for $x = \pm 8$.

$$x = 8 \quad \Rightarrow \quad y(8) = \sum_{n=1}^{\infty} 1 = 1 + 1 + 1 \cdots, \text{ diverges.}$$

$$x = 8 \quad \Rightarrow \quad y(-8) = \sum_{n=1}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \cdots, \text{ diverges.}$$

The interval of convergence is $I = (-8, 8)$. ◁

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Term by term derivation and integration

Theorem

If the power series $y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$ *has radius of convergence* $\rho > 0$, *then function* y *is both differentiable with derivative*

$$y'(x) = \sum_{n=1}^{\infty} n c_n (x - x_0)^{(n-1)},$$

and function y *is integrable with primitive*

$$\int y(x)\, dx = \sum_{n=0}^{\infty} \frac{(x - x_0)^{(n+1)}}{(n+1)} + c,$$

where both expressions above converge on $(x_0 - \rho, x_0 + \rho)$.