Alternating series and absolute convergence (Sect. 10.6)

- Alternating series.
- Absolute and conditional convergence.
- Absolute convergence test.
- Few examples.

Alternating series

Definition

An infinite series $\sum a_n$ is an *alternating series* iff holds either

$$a_n = (-1)^n \, |a_n| \quad \text{or} \quad a_n = (-1)^{n+1} \, |a_n|.$$

Example

- The alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

- The following series is an alternating series,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi) n^2}{(n+1)!} = \sum_{n=1}^{\infty} (-1)^n \, \frac{n^2}{(n+1)!} = -\frac{1}{2} + \frac{4}{6} - \frac{9}{24} + \cdots .$$

Alternating series

Theorem (Leibniz's test)

If the sequence $\{a_n\}$ satisfies: $0 < a_n$, and $a_{n+1} \leqslant a_n$, and $a_n \to 0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ converges.

Proof: Write down the partial sum s_{2n} as follows

$$\begin{aligned} s_{2n} &= a_1 - a_2 + a_3 - a_4 + a_5 - \cdots + s_{2n-1} - s_{2n} \\ &= (a_1 - a_2) + (a_3 - a_4) + \cdots + (s_{2n-1} - s_{2n}) \\ &= a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (s_{2n-2} - s_{2n-1}) - s_{2n}. \end{aligned}$$

The second expression implies $s_{2n} \leqslant s_{2(n+1)}$.

The third expression says that s_{2n} is bounded above.

Therefore converges, $s_{2n} \to L$.

Since $s_{2n+1} = s_{2n} + a_{2n+1}$, and $a_n \to 0$, then $s_{2n+1} \to L + 0 = L$.

We conclude that $\sum(-1)^{n+1}a_n$ converges. □

Alternating series

Example

Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$. converges.

Solution: Introduce the sequence $a_n = \dfrac{(-1)^{n+1}}{n}$.

The sequence $\{a_n\}$ satisfies the hypothesis in the Leibniz test:

- $|a_n| > 0$;
- $|a_{n+1}| < |a_n|$;
- $|a_n| \to 0$.

We then conclude that $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges. ◁

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Absolute and conditional convergence

Remarks:

- Several convergence tests apply only to positive series.
- Integral test, direct comparison and limit comparison tests, ratio test, do not apply to alternating series.
- Given an arbitrary series $\sum a_n$, the series $\sum |a_n|$ has non-negative terms.

Definition

- A series $\sum a_n$ is *absolutely convergent* iff the series $\sum |a_n|$ converges.
- A series *converges conditionally* iff it converges but does not converges absolutely.

Absolute and conditional convergence

Example

- The alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally.

  Because the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the alternating harmonic series converges.

- The geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2^n}$ converges absolutely.

  Because the geometric series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges.

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Absolute convergence test

Theorem

If the series $\sum |a_n|$ converges, then the series $\sum a_n$ converges.

Remark:

The converse is not true. A series can converge conditionally:

$\sum \frac{(-1)^{n+1}}{n}$ converges, but $\sum \left|\frac{(-1)^{n+1}}{n}\right|$ does not converge.

Proof: $-|a_n| \leqslant a_n \leqslant |a_n| \quad \Rightarrow \quad 0 \leqslant a_n + |a_n| \leqslant 2|a_n|$.

Since $\sum |a_n|$ converges so does $\sum 2|a_n|$.

Direct comparison test implies $\sum (a_n + |a_n|)$ converges.

$$\sum a_n = \sum (a_n + |a_n|) - \sum |a_n|,$$

and both series on the right-hand side converge.

Hence $\sum a_n$ converges. □

Alternating series and absolute convergence (Sect. 10.6)

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Few examples

Example

Determine whether the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{4n}{4n^6+5}$ converges absolutely, conditionally, or does not converge at all.

Solution: We first study absolute convergence. We use the ratio test in the sequence $a_n = \left|(-1)^{n+1}\frac{4n}{4n^6+5}\right| = \frac{4n}{4n^6+5}$.

$$\frac{a_{n+1}}{a_n} = \frac{4(n+1)}{[4(n+1)^6+5]}\,\frac{[4n^6+5]}{4n} = \frac{(n+1)}{n}\left[\frac{4n^6+5}{4(n+1)^6+5}\right] \to 1.$$

Ratio test inconclusive. Direct comparison test:

$$4n^6 < 4n^6+5 \;\Rightarrow\; \frac{1}{4n^6+5} < \frac{1}{4n^6} \;\Rightarrow\; \frac{4n}{4n^6+5} < \frac{1}{n^5}.$$

$\sum_{n=1}^{\infty}\frac{1}{n^5}$ converges, (IT), so the series converges absolutely. $\triangleleft$

Few examples

Example

Determine whether the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: The series diverges absolutely, since

$$|a_n| = \left|\frac{(-1)^{n+1}}{\ln(n)}\right| = \frac{1}{\ln(n)},$$

and $\ln(n) < n$ implies $\frac{1}{n} < \frac{1}{\ln(n)}$.

Since the harmonic series diverges, then $\sum_{n=1}^{\infty}\frac{1}{\ln(n)}$ diverges;

therefore, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\ln(n)}$ diverges absolutely.

Few examples

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges absolutely, conditionally, or does not converge at all.

Solution: Recall: The series diverges absolutely.

We now try the Leibniz test (the alternating series test)

$$|a_n| = \frac{1}{\ln(n)} > 0, \quad |a_n| = \frac{1}{\ln(n)} \to 0.$$

Furthermore, the inequality $\ln(n) < \ln(n+1)$ implies

$$|a_{n+1}| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = |a_n|.$$

Hence, the Leibniz test implies that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n)}$ converges.

Hence, the series converges conditionally. $\triangleleft$

Few examples

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^{n}}{n!}$ converges absolutely, conditionally, or does not converge at all.

Solution: We test absolute convergence: $|a_n| = \left|\dfrac{(-100)^n}{n!}\right| = \dfrac{100^n}{n!}$.

Let us check the ratio test:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{100^{n+1}}{(n+1)!}\,\frac{n!}{100^n} = \frac{100(100^n)}{(n+1)\,n!}\,\frac{n!}{100^n} = \frac{100}{(n+1)} \to 0.$$

The ratio test implies $\displaystyle\sum_{n=1}^{\infty} \frac{(-100)^{n}}{n!}$ coverges absolutely.

Therefore, the series converges. $\triangleleft$