Ratio test (Sect. 10.5)

- The ratio test.
- Using the ratio test.
- Few more examples.
- Comment: The root test.

The ratio test

Remark: The ratio test is a way to determine whether a series converges or not.

Theorem

Let $\{a_n\}$ be a positive sequence with $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$ exists.

(a) *If $\rho < 1$, the series $\sum a_n$ converges.*

(b) *If $\rho > 1$, the series $\sum a_n$ diverges.*

(c) *If $\rho = 1$, the test is inconclusive.*

Remark: The ratio test compares the series $\sum a_n$ with an appropriate geometric series $\sum r^n$.

The ratio test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists N large with

$$\frac{a_{n+1}}{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N.$$

$$\frac{a_{N+n}}{a_N} = \frac{a_{N+1}}{a_N}\,\frac{a_{N+2}}{a_{N+1}} \cdots \frac{a_{N+n}}{a_{N+n-1}} \leqslant r^n \quad \Rightarrow \quad a_{N+n} \leqslant a_N\, r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=0}^{\infty} a_{N+n} \leqslant \sum_{n=0}^{N-1} a_n + a_N \sum_{n=0}^{\infty} r^n$$

So $\displaystyle\sum_{n=0}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \frac{a_N}{1-r}$ is bounded.

A non-decreasing, bounded above, series converges.

The ratio test

Proof: Case (b): Since $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho > 1$, there exists N large,

$$\frac{a_{n+1}}{a_n} > 1, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_N < a_{N+1} < a_{N+2} < \cdots.$$

Hence, $\displaystyle\lim_{n\to\infty} a_n \neq 0$. The series $\sum a_n$ diverges.

Case (c): $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. Examples:

$$\sum_{n=1}^{\infty} \frac{1}{n}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{n+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1, \quad \text{diverges.}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \quad \text{and} \quad \lim_{n\to\infty} \frac{\left(\frac{1}{(n+1)^2}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1, \quad \text{converges.}$$

The test in inconclusive. $\square$

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Using the ratio test

Example

Determine whether the series $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{2^n}{n!} > 0$. We compute

$$\frac{a_{n+1}}{a_n} = \frac{2^{n+1}}{(n+1)!}\,\frac{n!}{2^n} = \frac{2^n\,2}{(n+1)\,n!}\,\frac{n!}{2^n} = \frac{2}{(n+1)}.$$

Therefore, $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2}{(n+1)} = 0.$

Since $\rho = 0 < 1$, the series converges. $\triangleleft$

Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n-1)!}{(n+1)^2}$ converges or not.

Solution: We use the ratio test, since $a_n = \frac{(n-1)!}{(n+1)^2} > 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{n!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n\,(n-1)!}{(n+2)^2}\,\frac{(n+1)^2}{(n-1)!} = \frac{n(n+1)^2}{(n+2)^2}$$

$$\frac{a_{n+1}}{a_n} = \frac{n^3+2n^2+n}{n^2+4n+4} = \frac{n+2+\frac{1}{n}}{1+\frac{4}{n}+\frac{4}{n^2}}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} n = \infty.$

Since $\rho = \infty > 1$, the series diverges. $\lhd$

Using the ratio test

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(n)}{n} \geqslant 0$. Then,

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{(n+1)}\,\frac{n}{\ln(n)} = \frac{n}{(n+1)}\,\frac{\ln(n+1)}{\ln(n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: $a_n = \dfrac{\ln(n)}{n} \geqslant \dfrac{1}{n}$ implies that

$$\sum \frac{\ln(n)}{n} \geqslant \sum \frac{1}{n}, \quad \text{which diverges.}$$

Therefore, the series diverges. $\lhd$

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Few more examples

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\ln(7n)}{n^3}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{\ln(7n)}{n^3} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{\ln[7(n+1)]}{(n+1)^3} \frac{n^3}{\ln(7n)} = \Big(\frac{n}{n+1}\Big)^3 \frac{\ln(7n+7)}{\ln(7n)} \to 1$$

Since $\rho = 1$, the ratio test is inconclusive.

Direct comparison test: Since $\ln(7n) < 7n$, then

$a_n = \dfrac{\ln(7n)}{n^3} \leqslant \dfrac{7n}{n^3} = \dfrac{7}{n^2}$. Hence $\displaystyle\sum \frac{\ln(7n)}{n^3} \leqslant \sum \frac{7}{n^2}$,

which converges. Therefore, the series converges. ◁

Few more examples

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{5n\ln(n)}{6^n}$ converges or not.

Solution: We start with the ratio test, since $a_n = \dfrac{5n\ln(n)}{6^n} \geqslant 0$.

$$\frac{a_{n+1}}{a_n} = \frac{5(n+1)\ln[(n+1)]}{6^{(n+1)}}\,\frac{6^n}{5n\ln(n)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{6}\Big(\frac{n+1}{n}\Big)\,\frac{\ln(n+1)}{\ln(n)} \to \frac{1}{6}$$

Since $\rho = \dfrac{1}{6} < 1$, the ratio test says that the series converges. $\triangleleft$

Ratio test (Sect. 10.5)

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Comment: The root test

Theorem

Let $\{a_n\}$ be a positive sequence with $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$ exists.

(a) *If $\rho < 1$, the series $\sum a_n$ converges.*

(b) *If $\rho > 1$, the series $\sum a_n$ diverges.*

(c) *If $\rho = 1$, the test is inconclusive.*

Remark: The root test also compares the series $\sum a_n$ with an appropriate geometric series $\sum r^n$.

Comment: The root test

Proof: Case (a): Since $a_n \geqslant 0$, the series $\sum a_n$ is non-decreasing. We now show that $\sum a_n$ is bounded above.

Since $\lim_{n\to\infty} \sqrt[n]{a_n} = \rho < 1$, then for any $\epsilon > 0$, small enough such that $\rho + \epsilon = r < 1$, there exists N large with

$$\sqrt[n]{a_n} < \rho + \epsilon = r, \quad \text{for } n \geqslant N, \quad \Rightarrow \quad a_n \leqslant r^n.$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n \leqslant \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} r^n$$

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n - \sum_{n=0}^{N-1} r^n + \sum_{n=0}^{\infty} r^n = \sum_{n=0}^{N-1} (a_n - r^n) + \frac{1}{1-r}.$$

So $\sum a_n$ is bounded. A non-decreasing, bounded above, series converges. The proofs for (b), (c) are similar to ratio test. $\square$