Comparison tests (Sect. 10.4)

- Review: Direct comparison test for integrals.
- Direct comparison test for series.
- Review: Limit comparison test for integrals.
- Limit comparison test for series.
- Few examples.

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Review: Direct comparison test for integrals

Theorem (Direct comparison test)

If $0 \leqslant \int_a^\infty f(x)\,dx \leqslant \int_a^\infty g(x)\,dx$, *then:*

(a) $\int_a^\infty g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^\infty f(x)\,dx$ *converges;*

(b) $\int_a^\infty f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^\infty g(x)\,dx$ *diverges.*

Example

$\int_0^\infty e^{-x^2}\,dx$ converges, since $\int_0^\infty e^{-x^2}\,dx \leqslant \int_0^\infty e^{-x}\,dx$.

$\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges, since $\int_2^\infty \frac{dx}{x} \leqslant \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$. $\triangleleft$

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Direct comparison test for series

Theorem

If the sequences satisfy $0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$, then

(a) $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} a_n$ *converges;*

(b) $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges* $\quad\Rightarrow\quad \displaystyle\sum_{n=1}^{\infty} b_n$ *converges.*

Example

Determine whether the the series $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ converges or not.

Solution: Since $\dfrac{n+2}{n^2-n} > \dfrac{n}{n^2-n} = \dfrac{1}{n-1} > \dfrac{1}{n}$, we conclude that:

$\displaystyle\sum_{n=2}^{\infty} \frac{1}{n} < \sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$. Therefore, $\displaystyle\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ diverges. $\quad\triangleleft$

Direct comparison test for series

Example

Determine whether the the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges or not.

Solution: For $1 \leqslant n$ holds, $\;3^n \leqslant n\,3^n \quad\Rightarrow\quad \dfrac{1}{n\,3^n} \leqslant \dfrac{1}{3^n}$.

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \Big(\frac{1}{3}\Big)^n = \sum_{n=0}^{\infty} \Big(\frac{1}{3}\Big)^n - 1,$$

$$\sum_{n=1}^{\infty} \frac{1}{n\,3^n} \leqslant \frac{1}{\big(1-\frac{1}{3}\big)} - 1 = \frac{1}{\big(\frac{3-1}{3}\big)} - 1 = \frac{3}{2} - 1 = \frac{1}{2}.$$

We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,3^n}$ converges. $\quad\triangleleft$

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Review: Limit comparison test for integrals

Theorem (Limit comparison test)

If positive functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$$

then either both $\int_a^\infty f(x)\,dx$, $\int_a^\infty g(x)\,dx$, *converge or diverge.*

Remark: If the integrals converge, their values may **not** agree.

Example

$\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges because $\int_1^\infty \frac{dx}{x^3}$ converges.

$\int_1^\infty \frac{dx}{\sqrt{x+\sin(x)}}$ diverges because $\int_1^\infty \frac{dx}{x^{1/2}}$ diverges.

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Limit comparison test for series

Theorem (Limit comparison test)

Assume that $0 < a_n$, and $0 < b_n$ for $N \leqslant n$.

(a) *If* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$, *then the infinite series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *both converge or both diverge.*

(b) *If* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*

(c) *If* $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, *and* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges.*

Remark: If the series converge, their values may **not** agree.

Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\frac{\sqrt{n}}{(4n^2+7)} = \frac{\sqrt{n}}{(4n^2+7)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{1}{n^{3/2}}\right)}{4 + \left(\frac{7}{n^2}\right)}$$

For n large $a_n = \dfrac{\sqrt{n}}{(4n^2+7)}$ behaves like $b_n = \dfrac{1}{4\,n^{3/2}}$.

We choose $b_n = \dfrac{1}{4\,n^{3/2}}$ to do the limit comparison test.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \left(\frac{\sqrt{n}}{(4n^2+7)}\right) 4n^{3/2} = \lim_{n\to\infty} \frac{4n^2}{(4n^2+7)} = 1.$$

Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges or not.

Solution: $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ both converge or diverge.

However, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges $\Leftrightarrow$ $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}}$ converges.

But: $\displaystyle\int_1^{\infty} \frac{dx}{4\,x^{3/2}} = \frac{1}{4}\,(-2)\,x^{-1/2}\Big|_1^{\infty} = \frac{1}{2}.$

Then, the integral test says that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{4\,n^{3/2}}$ converges.

The limit test for series says that $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{4n^2+7}$ converges. $\triangleleft$

Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: We compute the behavior of the series terms for n large:

$$\lim_{n\to\infty} \frac{3^{2n}}{2^n + n} = \lim_{n\to\infty} \frac{3^{2n}}{2^n} \quad \text{and} \quad \frac{3^{2n}}{2^n} = \frac{3^{2n}}{(\sqrt{2})^{2n}} = \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$$

For n large $a_n = \dfrac{3^{2n}}{(2^n + n)}$ behaves like $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$.

We choose $b_n = \Big(\dfrac{3}{\sqrt{2}}\Big)^{2n}$ to do the limit comparison test, hence

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 1$ and both $\displaystyle\sum_{n=1}^{\infty} a_n$, $\displaystyle\sum_{n=1}^{\infty} b_n$ converge or diverge.

Limit comparison test for series

Example

Determine whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ converges or not.

Solution: Both $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$, and $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$ converge or diverge.

Since $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$ is a geometric series with ratio $r = \dfrac{3}{\sqrt{2}} > 1$,

the series $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{3}{\sqrt{2}}\Big)^{2n}$ diverges.

We conclude that $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{2^n + n}$ diverges. $\lhd$

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Few examples

Example

(1) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2(n)}{6^n}$. **DGC** $\displaystyle\frac{\sin^2(n)}{6^n} \leqslant \Big(\frac{1}{6}\Big)^n$; $\displaystyle\sum_{n=1}^{\infty} \Big(\frac{1}{6}\Big)^n$ converges.

(2) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln(n)}$. **ID** $\displaystyle\int_3^{\infty} \frac{dx}{x\ln(x)} = \int_{\ln(3)}^{\infty} \frac{du}{u}$; $u = \ln(x)$.

Since $a_n = f(n)$ and $\displaystyle\int_3^{\infty} f(x)\,dx = \int_{\ln(3)}^{\infty} \frac{du}{u}$ diverges.

(3) $\displaystyle\sum_{n=1}^{\infty} \frac{n+5^n}{n^2 5^n}$. **LIC** $\displaystyle\frac{n+5^n}{n^2 5^n} \to \frac{5^n}{n^2 5^n} = \frac{1}{n^2}$;

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, since $\displaystyle\int_1^{\infty} \frac{dx}{x^2}$ converges. $\triangleleft$