The integral test (Sect. 10.3)

- Review: Bounded and monotonic sequences.
- Application: The harmonic series.
- Testing with an integral.
- Error estimation in the integral test.

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Review: Bounded and monotonic sequences

Definition

A sequence $\{a_n\}$ is bounded above iff there is $M \in \mathbb{R}$ such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1.$$

The sequence $\{a_n\}$ is bounded below iff there is $m \in \mathbb{R}$ such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1.$$

A sequence is bounded iff it is bounded above and below.

Example

- $a_n = \frac{1}{n}$ is bounded, since $0 < \frac{1}{n} \leqslant 1$.
- $a_n = (-1)^n$ is bounded, since $-1 \leqslant (-1)^n \leqslant 1$.

Review: Bounded and monotonic sequences

Definition

- A sequence $\{a_n\}$ is increasing iff $a_n < a_{n+1}$.
- A sequence $\{a_n\}$ is non-decreasing iff $a_n \leqslant a_{n+1}$.
- A sequence $\{a_n\}$ is decreasing iff $a_n > a_{n+1}$.
- A sequence $\{a_n\}$ is non-increasing iff $a_n \geqslant a_{n+1}$.
- A sequence is monotonic iff the sequence is both non-increasing and non-decreasing.

Theorem

- *A non-decreasing sequence converges iff it is bounded above.*
- *A non-increasing sequence converges iff it bounded below.*

Review: Bounded and monotonic sequences

Example

Determine whether the sequence $a_n = \dfrac{n}{n^2+1}$ converges or not.

Solution: We show that a_n is decreasing. Indeed, the condition

$$a_{n+1} < a_n \quad \Leftrightarrow \quad \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) < n(n^2+2n+2)$$

$$n^3+n^2+n+1 < n^3+2n^2+2n$$

Since $1 < (n^2+n)$ is true for $n \geqslant 1$, then $a_{n+1} < a_n$; decreasing.

The sequence satisfies that $0 < a_n$, bounded below.

We conclude that a_n converges. $\triangleleft$

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Application: The harmonic series

Remarks:

- The partial sums of the harmonic series, $s_n = \sum_{k=1}^{n} \frac{1}{k}$, define an increasing sequence: $s_{n+1} > s_n$.
- We now show that $\{s_n\}$ is unbounded from above.

Example

Show that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: Notice the following inequalities:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left[\frac{1}{3} + \frac{1}{4}\right] + \left[\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right] + \cdots$$

$$\sum_{n=1}^{\infty} \frac{1}{n} > 1 + \frac{1}{2} + \left[\frac{2}{4}\right] + \left[\frac{4}{8}\right] + \cdots \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges.} \quad \triangleleft$$

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Testing with an integral

Remark:

- The idea used above to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges cannot be generalized to other series.
- Now we introduce an idea to test the convergence of series. The idea is based on calculus.

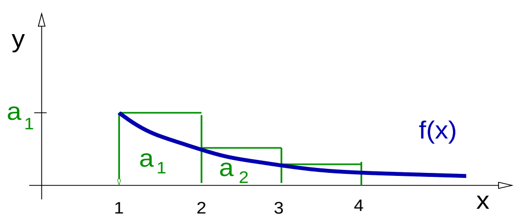
Theorem

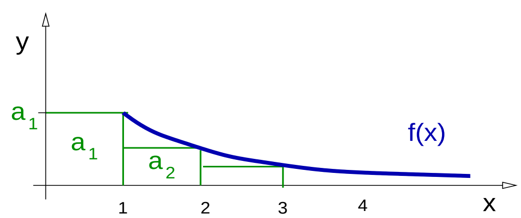
If $f : [1, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, and $a_n = f(n)$, then the following holds:

$$\sum_{n=1}^{\infty} a_n \quad \text{converges} \quad \Leftrightarrow \quad \int_1^{\infty} f(x)\,dx \quad \text{converges.}$$

Testing with an integral

Proof: Recall: $a_n = f(n)$. The proof is based in the pictures:





$$\int_1^4 f(x)\,dx \leqslant a_1 + a_2 + a_3 \quad \Rightarrow \quad \int_1^{n+1} f(x)\,dx \leqslant \sum_{k=1}^{n} a_k.$$

$$a_2 + a_3 \leqslant \int_1^3 f(x)\,dx \quad \Rightarrow \quad \sum_{k=1}^{n} a_k \leqslant a_1 + \int_1^n f(x)\,dx.$$

$$\int_1^{n+1} f(x)\,dx \;\leqslant\; \sum_{k=1}^{n} a_n \;\leqslant\; a_1 + \int_1^n f(x)\,dx. \qquad \square$$

Testing with an integral

Example

Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution: The convergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is related to the convergence of the integral $\int_1^{\infty} \frac{dx}{x}$. Since

$$\ln(n+1) = \int_1^{n+1} \frac{dx}{x} \leqslant \sum_{k=1}^{n} a_n \quad \text{and} \quad \ln(n+1) \to \infty$$

then the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. ◁

Testing with an integral

Example

Show whether the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ converges or not.

Solution: The convergence of the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2}$ is related to the convergence of the integral $\int_1^{\infty} \frac{dx}{1+x^2}$. Since

$$\int_1^{n} \frac{dx}{1+x^2} = \arctan(x)\big|_1^n = \Big(\arctan(n) - \frac{\pi}{4}\Big) \to \Big(\frac{\pi}{2} - \frac{\pi}{4}\Big).$$

The inequality $\sum_{k=1}^{\infty} a_k \leqslant a_1 + \int_1^{\infty} f(x)\, dx$ implies

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} \leqslant \frac{1}{2} + \frac{\pi}{4} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{1+n^2} \text{ converges.}$$

◁

Testing with an integral

Example

Show whether the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}$ converges or not.

Solution: The convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}$ is related to the convergence of the integral $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}\sqrt{x+1}}$.

Limit test for improper integrals: $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{x}\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{x}$.

Since $\displaystyle\int_1^{\infty} \frac{dx}{x}$ diverges, then $\displaystyle\int_1^{\infty} \frac{dx}{\sqrt{x}\sqrt{x+1}}$ diverges.

Integral test for series implies: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\sqrt{n+1}}$ diverges. $\triangleleft$

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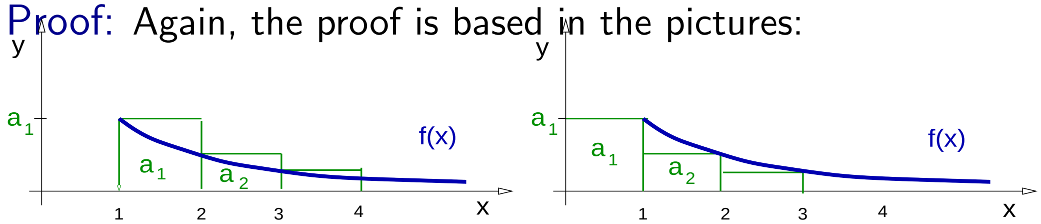
Error estimation in the integral test.

Theorem

If $f : [1, \infty) \to \mathbb{R}$ is a continuous, positive, decreasing function, and the series $\sum_{k=1}^{n} a_k = s_n \to S$, where $a_n = f(n)$, then the remainder $R_n = S - s_n$ satisfies

$$\int_{n+1}^{\infty} f(x)\,dx \leqslant R_n \leqslant \int_{n}^{\infty} f(x)\,dx.$$

Proof: Again, the proof is based in the pictures:



$$\int_{3}^{\infty} f(x)\,dx \leqslant R_2 = a_3 + a_4 + \cdots \leqslant \int_{2}^{\infty} f(x)\,dx \qquad \square$$