Infinite series (Sect. 10.2)

- Series and partial sums.
- Geometric series.
- The *n*-term test for a divergent series.
- Operations with series.
- Adding-deleting terms and re-indexing.

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Series and partial sums

Definition

An infinite series is a sum of infinite terms,

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n.$$

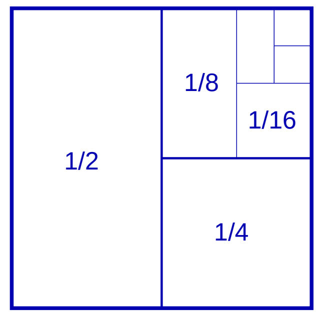
Remark: Any sequence $\{a_n\}_{n=1}^{\infty}$ defines the series $\displaystyle\sum_{n=1}^{\infty} a_n$.

Example

The sequence $\left\{a_n = \dfrac{1}{2^n}\right\}_{n=1}^{\infty}$ defines the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots$$

This infinite sum makes sense, since



Series and partial sums

Definition

Given an infinite series $\displaystyle\sum_{n=1}^{\infty} a_n$, the sequence of partial sums of the series is the sequence $\{s_n\}$ given by $s_n = \displaystyle\sum_{k=1}^{n} a_k$, that is,

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \end{aligned}$$

The series converges to L iff the sequence of partial sums $\{s_n\}$ converges to L, and in this case we write $\displaystyle\sum_{n=1}^{\infty} a_n = L$. The series diverges iff the sequence of partial sums $\{s_n\}$ diverges.

Series and partial sums

Remark: The series $a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ can be denoted as

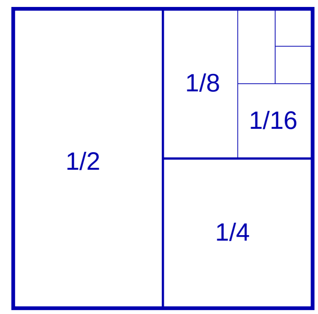
$$\sum_{n=1}^{\infty} a_n, \qquad \sum_{k=1}^{\infty} a_k, \qquad \sum a_n$$

Example

The series $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to 1,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Since $\{s_n\} \to 1$, as can be seen in the picture.



Series and partial sums

Example

- The series $\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \cdots + n + \cdots$ diverges.
  Indeed, the sequence of partial sums diverges,

$$s_1 = 1, \quad s_2 = 3, \quad s_3 = 6, \quad s_n = \sum_{k=1}^{n} k.$$

- The series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$ is called the harmonic series. We will see that the harmonic series diverges.

- While the series $\sum_{n=1}^{\infty} \frac{(-1)^{(n+1)}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges.

Series and partial sums

Example

Evaluate the infinite series $\frac{1}{2} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \frac{1}{(4)(5)} + \cdots$.

Solution: We first find the general term a_n, that is,

$$a_n = \frac{1}{n(n+1)}, \qquad n = 1, \cdots \infty.$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{(2)(3)} + \frac{1}{(3)(4)} + \frac{1}{(4)(5)} + \cdots$$

Partial fractions implies $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \Big(\frac{1}{n} - \frac{1}{(n+1)}\Big)$. So,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \cdots = 1.$$

We conclude: $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$ ◁

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Geometric series

Definition

A geometric series is a series of the form

$$\sum_{n=0}^{\infty} a\, r^n = a + a\, r + a\, r^2 + a\, r^3 + \cdots .$$

where a and r are real numbers.

Example

The case $a = 1$, and ratio $r = \frac{1}{2}$ is the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots .$$

We have seen $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$, so $\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 2$

Geometric series

Theorem

If the geometric series $\sum_{n=0}^{\infty} a\, r^n$ *has ratio* $|r| < 1$, *then converges,*

$$\sum_{n=0}^{\infty} a\, r^n = \frac{a}{1 - r}.$$

Proof: Multiply any partial sum s_n by $(1 - r)$, that is,

$$(1 - r)s_n = a(1 - r)(1 + r + r^2 + \cdots + r^n)$$

$$(1 - r)s_n = a(1 + r + r^2 + \cdots + r^n) - a(r + r^2 + r^3 + \cdots + r^{n+1})$$

$$(1 - r)s_n = a(1 - r^{n+1}) \quad \Rightarrow \quad s_n = \frac{a(1 - r^{n+1})}{(1 - r)}.$$

Since $|r| < 1$, then $r^{n+1} \to 0$. $\square$

Geometric series

Example

Evaluate the infinite series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n}$.

Solution: Recall the picture says $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$.

We now use the Theorem above, $\displaystyle\sum_{n=0}^{\infty} a\, r^n = \frac{a}{1-r}$,

for $a = 1$ and $r = \dfrac{1}{2}$.

$$\sum_{n=0}^{\infty} \Big(\frac{1}{2}\Big)^n = \frac{1}{1-\frac{1}{2}} = \frac{1}{\left(\frac{2-1}{2}\right)} \quad \Rightarrow \quad \sum_{n=0}^{\infty} \Big(\frac{1}{2}\Big)^n = 2. \qquad \triangleleft$$

Geometric series

Example

Evaluate the infinite sum $\displaystyle\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{3}{4^n}$.

Solution: This is a geometric series, since

$$\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{3}{4^n} = \sum_{n=1}^{\infty} (-1)^n \frac{(-3)}{4^n} = \sum_{n=1}^{\infty} (-3) \left(-\frac{1}{4}\right)^n.$$

Hence $a = -3$ and $r = -\dfrac{1}{4}$. The Theorem above implies,

$$\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{3}{4^n} = \sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{3}{4^n} - (-3) = \frac{(-3)}{\left(1+\frac{1}{4}\right)} + 3$$

$$\sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{3}{4^n} = -\frac{3}{\left(\frac{4+1}{4}\right)} + 3, \text{ then } \sum_{n=1}^{\infty} (-1)^{(n+1)} \frac{3}{4^n} = -\frac{12}{5} + 3.$$

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The n-term test for a divergent series

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *converges, then* $a_n \to 0$.

Remark: This result is useful to find divergent series.

Remark: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example

- $\sum_{n=1}^{\infty} n$ diverges, since $n \to \infty$.
- $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges, since $\frac{n}{n+1} \to 1 \neq 0$.
- $\sum_{n=1}^{\infty} (-1)^n$ diverges, since $\lim_{n\to\infty} (-1)^n$ does not exist.

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Operations with series

Remark: Additions of convergent series define convergent series.

Theorem

If the series $\displaystyle\sum_{n=1}^{\infty} a_n = A$ *and* $\displaystyle\sum_{n=1}^{\infty} b_n = B$*, then*

- $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = A + B;$
- $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = A - B;$
- $\displaystyle\sum_{n=1}^{\infty} k a_n = kA.$

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Adding-deleting terms and re-indexing

Remarks:

- Adding or deleting a finite number of terms to series does not change the series convergence or divergence.

Example: $$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \sum_{n=4}^{\infty} \frac{1}{2^n}.$$

- The same series can be written using different indexes.

Example: $$\sum_{n=1}^{\infty} a_n = \sum_{\ell=1}^{\infty} a_\ell = \sum_{k=7}^{\infty} a_{k-6}.$$

Example: $$\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{k=8}^{\infty} \frac{1}{2^{(k-7)}} = \sum_{k=8}^{\infty} \frac{2^7}{2^k}.$$