Review for Exam 3.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 8.3, 8.4, 7.5, 8.7, 10.1.
  - Trigonometric substitutions (8.3).
  - Integration using partial fractions (8.4).
  - L'Hôpital's rule (7.5).
  - Improper integrals (8.7).
  - Infinite sequences (10.1).
- Section not covered:
  - Integration using tables (8.5).

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Trigonometric substitutions (8.3)

Recall: From Sect. 8.2: $\int \sec(\theta)\, d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c.$

Example

Evaluate $I = \int \frac{e^x\, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Solution: First substitution, $y = e^x$, then $dy = e^x\, dx$, $y > 0$,

$$I = \int \frac{dy}{\sqrt{y^2 + 9}}. \quad \text{Second subs.:} \quad \begin{cases} y = 3\tan(\theta), \\ dy = 3\sec^2(\theta)\, d\theta, \\ \theta \in (0, \pi/2). \end{cases}$$

$$I = \int \frac{3\sec^2(\theta)\, d\theta}{\sqrt{9\tan^2(\theta) + 9}} = \int \frac{3\sec^2(\theta)\, d\theta}{3\sqrt{\tan^2(\theta) + 1}} = \int \frac{\sec^2(\theta)\, d\theta}{|\sec(\theta)|}.$$

Trigonometric substitutions (8.3)

Example

Evaluate $I = \int \frac{e^x\, dx}{\sqrt{e^{2x} + 9}}$. Express you result in terms of the variable x.

Solution: So: $I = \int \frac{\sec^2(\theta)\, d\theta}{|\sec(\theta)|}$; $e^x = y = 3\tan(\theta)$; $\theta \in \left(0, \frac{\pi}{2}\right)$.

$$I = \int \sec(\theta)\, d\theta = \ln\big(\sec(\theta) + \tan(\theta)\big) + c$$

Recall, $\tan(\theta) = \frac{e^x}{3}$, hence $\sec(\theta) = \sqrt{\tan^2(\theta) + 1} = \sqrt{\frac{e^{2x}}{9} + 1}$.

We conclude that,

$$I = \ln\left(e^x + \frac{1}{3}\sqrt{e^{2x} + 9}\right) + c.$$

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Review for Exam 3.

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Integration using partial fractions (8.4)

Recall: If the polynomial in the numerator has larger degree than the polynomial in the denominator, then do the long division first.

Example

Evaluate $I = \int \frac{(x-2)}{(x^2 - x - 6)}\, dx.$

Solution: We find the roots of the denominator, $x^2 - x - 6 = 0$,

$$x_\pm = \frac{1}{2}\big(1 \pm \sqrt{1+24}\big) = \frac{1}{2}(1 \pm 5) \quad \Rightarrow \quad \begin{cases} x_+ = 3, \\ x_- = -2. \end{cases}$$

Therefore, $I = \int \frac{(x-2)}{(x-3)(x+2)}\, dx$. Now, partial fractions:

$$\frac{(x-2)}{(x-3)(x+2)} = \frac{a}{(x-3)} + \frac{b}{(x+2)} \;\Rightarrow\; x-2 = a(x+2) + b(x-3).$$

Integration using partial fractions (8.4)

Example

Evaluate $I = \int \frac{(x-2)}{(x^2-x-6)}\,dx$.

Solution: Recall that:

$$I = \int \Big[\frac{a}{(x-3)} + \frac{b}{(x+2)}\Big]\,dx; \quad x-2 = a(x+2) + b(x-3).$$

Evaluating at $x = 3$ we get $a = \frac{1}{5}$, and at $x = -2$ we get $b = \frac{4}{5}$.

$$I = \frac{1}{5}\int \Big[\frac{1}{(x-3)} + \frac{4}{(x+2)}\Big]\,dx = \frac{1}{5}\big(\ln|x-3| + 4\ln|x+2|\big) + c.$$

We conclude that $I = \ln\big(|x-3|^{1/5}(x+2)^{4/5}\big) + c$. $\triangleleft$

Integration using partial fractions (8.4)

Remark: Incomplete summary of partial fraction decompositions:

- $\frac{p_2(x)}{(x-r_1)(x-r_2)(x-r_3)} = \frac{c_1}{(x-r_1)} + \frac{c_2}{(x-r_2)} + \frac{c_3}{(x-r_3)}$.
- $\frac{p_2(x)}{(x-r_1)^3} = \frac{c_1}{(x-r_1)} + \frac{c_2}{(x-r_1)^2} + \frac{c_3}{(x-r_1)^3}$.
- $\frac{p_2(x)}{(x-r_1)(x-r_2)^2} = \frac{c_1}{(x-r_1)} + \frac{c_2}{(x-r_2)} + \frac{(c_3x+c_4)}{(x-r_2)^2}$.
- $\frac{p_5(x)}{(x^2+b^2)^3} = \frac{(c_1x+c_2)}{(x^2+b^2)} + \frac{(c_3x+c_4)}{(x^2+b^2)^2} + \frac{(c_5x+c_6)}{(x^2+b^2)^3}$.
- $\frac{p_4(x)}{(x-r_1)(x^2+b^2)^2} = \frac{c_1}{(x-r_1)} + \frac{(c_2x+c_3)}{(x^2+b^2)} + \frac{(c_4x+c_5)}{(x^2+b^2)^2}$.

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L'Hôpital's rule (7.5)

Example

Evaluate the limit $L = \lim_{x\to\infty}\Big(1 - \frac{2}{x} - \frac{3}{x^2}\Big)^{8x}$.

Solution: We first rewrite the limit as follows,

$$L = \lim_{x\to\infty}\Big(1 - \frac{2}{x} - \frac{3}{x^2}\Big)^{8x} = \lim_{x\to\infty} e^{\left[8x\,\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)\right]}$$

$$L = e^{\lim_{x\to\infty}\left[8x\,\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)\right]} = e^{\lim_{x\to\infty}\left[\frac{\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{8x}}\right]}$$

L'Hôpital rule in the exponent implies,

$$\lim_{x\to\infty}\frac{\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{8x}} = \lim_{x\to\infty}\frac{\left(1-\frac{2}{x}-\frac{3}{x^2}\right)^{-1}\left(\frac{2}{x^2}+\frac{6}{x^3}\right)}{\left(-\frac{1}{8x^2}\right)}$$

L'Hôpital's rule (7.5)

Example

Evaluate the limit $L = \lim_{x\to\infty} \left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{8x}$.

Solution: Recall: $L = e^{\lim_{x\to\infty} \left[\frac{\ln\left(1-\frac{2}{x}-\frac{3}{x^2}\right)}{\frac{1}{8x}}\right]}$, and

$$\tilde{L} = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)}{\frac{1}{8x}} = \lim_{x\to\infty} \frac{\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{-1}\left(\frac{2}{x^2} + \frac{6}{x^3}\right)}{\left(-\frac{1}{8x^2}\right)}$$

$$\tilde{L} = \lim_{x\to\infty} -8\left(1 - \frac{2}{x} - \frac{3}{x^2}\right)^{-1}\left(2 + \frac{6}{x}\right) = -16$$

We conclude that $L = e^{-16}$. $\triangleleft$

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Improper integrals (8.7)

Example

Evaluate the integral $I = \int_0^5 \frac{(x+1)}{\sqrt{25-x^2}}\,dx.$

Solution: We split the integral in two terms,

$$I = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx + \int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx.$$

On the first term: $u = 25 - x^2$, $du = -2x\,dx$. Hence,

$$I_1 = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx = \int_{25}^0 -\frac{1}{\sqrt{u}}\,\frac{du}{2} = \frac{1}{2}\int_0^{25} u^{-1/2}\,du.$$

$$I_1 = \frac{1}{2}\lim_{c\to 0^+}\int_c^{25} u^{-1/2}\,du = \frac{1}{2}\lim_{c\to 0^+} 2u^{1/2}\Big|_c^{25} \quad \Rightarrow \quad I_1 = 5.$$

Improper integrals (8.7)

Example

Evaluate the integral $I = \int_0^5 \frac{(x+1)}{\sqrt{25-x^2}}\,dx.$

Solution: Recall: $I_1 = \int_0^5 \frac{x}{\sqrt{25-x^2}}\,dx = 5.$

In the second integral: $x = 5\sin(\theta)$, $dx = 5\cos(\theta)\,d\theta$; Hence

$$I_2 = \int_0^5 \frac{dx}{\sqrt{25-x^2}} = \int_0^{\pi/2} \frac{5\cos(\theta)\,d\theta}{\sqrt{25-25\sin^2(\theta)}}$$

$$I_2 = \int_0^{\pi/2} \frac{\cos(\theta)}{|\cos(\theta)|}\,d\theta = \int_0^{\pi/2} d\theta \quad \Rightarrow \quad I_2 = \frac{\pi}{2}.$$

We conclude that $I = 5 + \frac{\pi}{2}$. ◁

Improper integrals (8.7): Comparison tests

- Direct comparison test:

  If $0 \leqslant f(x) \leqslant g(x)$ for $x \in [a, \infty)$, then holds
  $$0 \leqslant \int_a^{\infty} f(x)\,dx \leqslant \int_a^{\infty} g(x)\,dx.$$
  (a) $\int_a^{\infty} g(x)\,dx$ converges $\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ converges;

  (b) $\int_a^{\infty} f(x)\,dx$ diverges $\Rightarrow$ $\int_a^{\infty} g(x)\,dx$ diverges.

- Limit comparison test:

  If $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, with $0 < L < \infty$, then the integrals
  $$\int_a^{\infty} f(x)\,dx, \qquad \int_a^{\infty} g(x)\,dx$$
  both converge or both diverge.

Improper integrals (8.7): Comparison tests

Example

Determine whether $I = \int_3^{\infty} \frac{x\,dx}{\sqrt{x^5 + x^3}}$ converges or not.

Solution: First, find an appropriate function $g(x)$ such that:
$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^5 + x^3}} = \lim_{x\to\infty} \frac{x}{x^{5/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}}.$$
Therefore, we use the limit comparison test with $g(x) = x^{-3/2}$.
Then, by construction,
$$\lim_{x\to\infty} \Big(\frac{x}{\sqrt{x^5 + x^3}}\Big)\Big(\frac{1}{x^{-3/2}}\Big) = \lim_{x\to\infty} \Big(\frac{x}{x^{5/2}}\Big)\Big(\frac{1}{x^{-3/2}}\Big) = 1.$$
Since $\int_3^{\infty} x^{-3/2}\,dx = -2\,x^{-1/2}\Big|_3^{\infty} = -2\Big(0 - \frac{1}{\sqrt{3}}\Big) = \frac{2}{\sqrt{3}}$,
we conclude that I converges. ◁

Improper integrals (8.7): Comparison tests

Example

Determine whether $I = \int_3^\infty \frac{x\,dx}{\sqrt{x^5 + x^3}}$ converges or not.

Solution: We use the Direct Comparison Test: For $x > 0$ holds

$$x^5 < x^5 + x^3 \quad \Rightarrow \quad \frac{1}{x^5 + x^3} < \frac{1}{x^5}$$

$$\frac{1}{\sqrt{x^5 + x^3}} < \frac{1}{\sqrt{x^5}} = x^{-5/2} \;\Rightarrow\; \frac{x}{\sqrt{x^5 + x^3}} < x^{-5/2} x = x^{-3/2}.$$

$$I < \int_3^\infty x^{-3/2}\,dx = -2\,x^{-1/2}\Big|_3^\infty = -2\Big(0 - \frac{1}{\sqrt{3}}\Big) = \frac{2}{\sqrt{3}}.$$

We conclude that I converges. $\triangleleft$

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Infinite sequences (10.1)

Example

Evaluate $L = \lim_{n\to\infty} \left(\frac{8}{3n}\right)^{\frac{1}{3n}}$.

Solution: We study a similar limit for the function $f(x) = \left(\frac{8}{3x}\right)^{\frac{1}{3x}}$.

$$\lim_{x\to\infty} \left(\frac{8}{3x}\right)^{\frac{1}{3x}} = \lim_{x\to\infty} e^{\left[\frac{\ln\left(\frac{8}{3x}\right)}{3x}\right]} = e^{\lim_{x\to\infty}\left[\frac{\ln\left(\frac{8}{3x}\right)}{3x}\right]}$$

Now, L'Hôpital's rule to find the limit in the exponent;

$$\tilde{L} = \lim_{x\to\infty} \frac{\ln\left(\frac{8}{3x}\right)}{3x} = \lim_{x\to\infty} \frac{\left(\frac{3x}{8}\,\frac{(-8)}{3x^2}\right)}{3} = \lim_{x\to\infty} -\frac{1}{3x} = 0.$$

Hence, $\lim_{x\to\infty} f(x) = e^0 = 1$, therefore, $\lim_{n\to\infty} \left(\frac{8}{3n}\right)^{\frac{1}{3n}} = 1$. $\lhd$