Infinite sequences (Sect. 10.1)

Today's Lecture:

- Review: Infinite sequences.
- The Continuous Function Theorem for sequences.
- Using L'Hôpital's rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

Previous Lecture:

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- The Sandwich Theorem for sequences.

Review: Infinite sequences

Definition

An infinite sequence of numbers is an ordered set of real numbers.

Definition

An infinite sequence $\{a_n\}$ has limit L iff for every number $\epsilon > 0$ there exists a positive integer N such that

$$N < n \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

A sequence is called convergent iff it has a limit, otherwise it is called divergent.

Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.

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The Continuous Function Theorem for sequences

Theorem

If a sequence $\{a_n\} \to L$ and a continuous function f is defined both at L and every a_n, then the sequence $\{f(a_n)\} \to f(L)$.

Example

Find the limit of $\Big\{\ln\Big(\frac{(2+n+3n^2)}{(2n^2+3)}\Big)\Big\}$ as $n \to \infty$.

Solution: The sequence $b_n = \ln\Big(\frac{(2+n+3n^2)}{(2n^2+3)}\Big)$ can be written as

$$b_n = f(a_n), \quad f(x) = \ln(x), \quad a_n = \frac{(2+n+3n^2)}{(2n^2+3)}.$$

$$a_n = \frac{(2+n+3n^2)}{(2n^2+3)} \, \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{2}{n^2}+\frac{1}{n}+3\right)}{\left(2+\frac{3}{n^2}\right)} \to \frac{3}{2}.$$

We conclude that $b_n \to \ln\Big(\frac{3}{2}\Big)$. ◁

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Using L'Hôpital's rule on sequences

Theorem (L'Hôpital's rule for sequences)

If the sequence $\{a_n\}$ satisfies that:

- *There exist a function f such that for $n > N$ the sequence elements a_n can be written as $a_n = f(n)$;*
- *And $\lim_{x\to\infty} f(x) = L$;*

then holds that $\lim_{n\to\infty} a_n = L$.

Remark: The $\lim_{x\to\infty} f(x)$ may indeterminate, and *L'Hôpital's rule* might be used to compute that limit.

Example

Find the limit $a_n = \sqrt[8n]{5n}$ as $n \to \infty$.

Solution: Notice that $a_n = f(n)$ for $f(x) = \sqrt[8x]{5x}$.

Using L'Hôpital's rule on sequences

Example

Find the limit $a_n = \sqrt[8n]{5n}$ as $n \to \infty$.

Solution: Recall: $a_n = f(n)$ for $f(x) = \sqrt[8x]{5x}$.

$$\sqrt[8x]{5x} = e^{\ln(\sqrt[8x]{5x})} = e^{\left(\frac{\ln(5x)}{8x}\right)}$$

But $\lim_{x\to\infty} \dfrac{\ln(5x)}{8x}$ is indeterminate $\dfrac{\infty}{\infty}$. L'Hôpital's rule,

$$\lim_{x\to\infty} \frac{\ln(5x)}{8x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{8} = \lim_{x\to\infty} \frac{1}{8x} = 0.$$

$$\lim_{x\to\infty} \sqrt[8x]{5x} = \lim_{x\to\infty} e^{\left(\frac{\ln(5x)}{8x}\right)} = e^0 \quad \Rightarrow \quad \lim_{x\to\infty} \sqrt[8x]{5x} = 1.$$

We conclude that $\sqrt[8n]{5n} \to 1$ as $n \to \infty$. ◁

Using L'Hôpital's rule on sequences

Example

Given positive numbers a, b, find the $\lim_{n\to\infty} \left(1 - \dfrac{b}{n}\right)^{an}$.

Solution: We rewrite the sequence as follows,

$$\left(1 - \frac{b}{n}\right)^{an} = e^{\left[an\ln(1-\frac{b}{n})\right]} = e^{\left[\frac{a\ln(1-\frac{b}{n})}{\frac{1}{n}}\right]}$$

The exponent has an indeterminate limit, $\dfrac{a\ln(1-\frac{b}{n})}{\frac{1}{n}} \to \dfrac{0}{0}$.

Recall the argument with the L'Hôpital's rule on functions,

$$\lim_{x\to\infty} \frac{a\ln(1-\frac{b}{x})}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{a}{(1-\frac{b}{x})}\,\frac{b}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{ab}{(1-\frac{b}{x})} = ab.$$

We conclude that $\lim_{n\to\infty} \left(1 - \dfrac{b}{n}\right)^{an} = e^{ab}$. ◁

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Table of useful limits

Remark: The following limits appear often in applications:

- $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0;$
- $\lim_{n\to\infty} \sqrt[n]{n} = 1;$
- $\lim_{n\to\infty} x^{\left(\frac{1}{n}\right)} = 1, \quad \text{for} \quad x > 0;$
- $\lim_{n\to\infty} x^n = 0, \quad \text{for} \quad |x| < 1;$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x, \quad \text{for } x \in \mathbb{R};$
- $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

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Bounded and monotonic sequences

Definition

A sequence $\{a_n\}$ is bounded above iff there is $M \in \mathbb{R}$ such that

$$a_n \leqslant M \qquad \text{for all } n \geqslant 1.$$

The sequence $\{a_n\}$ is bounded below iff there is $m \in \mathbb{R}$ such that

$$m \leqslant a_n \qquad \text{for all } n \geqslant 1.$$

A sequence is bounded iff it is bounded above and below.

Example

- $a_n = \dfrac{1}{n}$ is bounded, since $0 < \dfrac{1}{n} \leqslant 1$.
- $a_n = (-1)^n$ is bounded, since $-1 \leqslant (-1)^n \leqslant 1$.

Bounded and monotonic sequences

Definition

- A sequence $\{a_n\}$ is increasing iff $a_n < a_{n+1}$.
- A sequence $\{a_n\}$ is non-decreasing iff $a_n \leqslant a_{n+1}$.
- A sequence $\{a_n\}$ is decreasing iff $a_n > a_{n+1}$.
- A sequence $\{a_n\}$ is non-increasing iff $a_n \geqslant a_{n+1}$.
- A sequence is monotonic iff the sequence is both non-increasing and non-decreasing.

Theorem

- *A non-decreasing, bounded above sequence, converges.*
- *A non-increasing, bounded below sequence, converges.*

Bounded and monotonic sequences

Example

Determine whether the sequence $a_n = \dfrac{n}{n^2+1}$ converges or not.

Solution: We show that a_n is decreasing. Indeed, the condition

$$a_{n+1} < a_n \quad \Leftrightarrow \quad \frac{n+1}{(n+1)^2+1} < \frac{n}{n^2+1}$$

$$(n+1)(n^2+1) < n(n^2+2n+2)$$

$$n^3 + n^2 + n + 1 < n^3 + 2n^2 + 2n$$

Since $1 < (n^2+n)$ is true for $n \geqslant 1$, then $a_{n+1} < a_n$; decreasing.

The sequence satisfies that $0 < a_n$, bounded below.

We conclude that a_n converges. $\triangleleft$