# Infinite sequences (Sect. 10.1)

### Today's Lecture:

- ► Review: Infinite sequences.
- ► The Continuous Function Theorem for sequences.
- ► Using L'Hôpital's rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

### Previous Lecture:

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- ► The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- ► The Sandwich Theorem for sequences.

# Review: Infinite sequences

### Definition

An infinite sequence of numbers is an ordered set of real numbers.

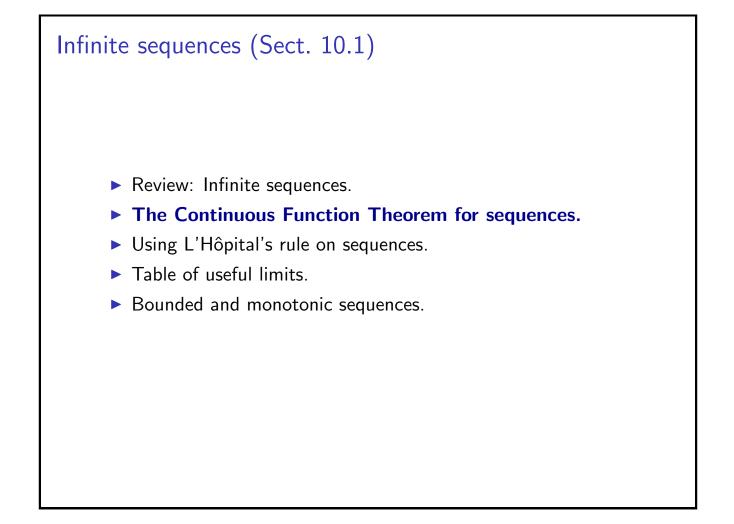
### Definition

An infinite sequence  $\{a_n\}$  has limit *L* iff for every number  $\epsilon > 0$  there exists a positive integer *N* such that

 $N < n \quad \Rightarrow \quad |a_n - L| < \epsilon.$ 

A sequence is called convergent iff it has a limit, otherwise it is called divergent.

Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.



## The Continuous Function Theorem for sequences

#### Theorem

If a sequence  $\{a_n\} \to L$  and a continuous function f is defined both at L and every  $a_n$ , then the sequence  $\{f(a_n)\} \to f(L)$ .

Example

Find the limit of  $\left\{ \ln\left(\frac{(2+n+3n^2)}{(2n^2+3)}\right) \right\}$  as  $n \to \infty$ .

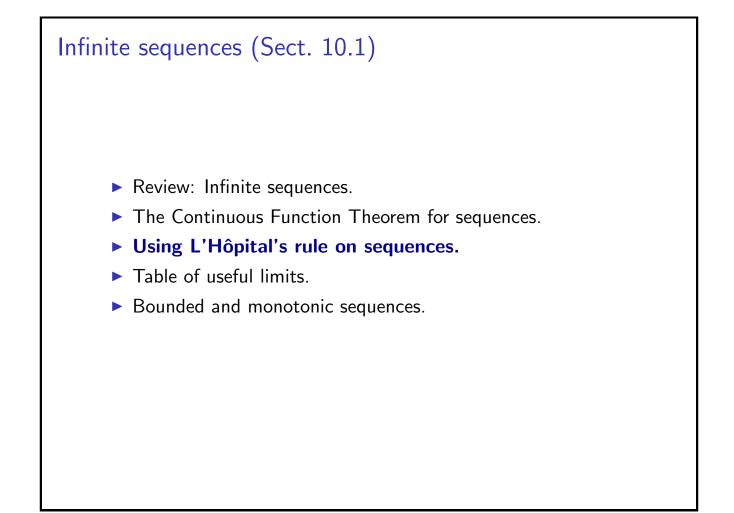
Solution: The sequence  $b_n = \ln\left(\frac{(2+n+3n^2)}{(2n^2+3)}\right)$  can be written as

$$b_n = f(a_n), \quad f(x) = \ln(x), \quad a_n = \frac{(2+n+3n^2)}{(2n^2+3)},$$

$$a_n = \frac{(2+n+3n^2)}{(2n^2+3)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{\left(\frac{2}{n^2}+\frac{1}{n}+3\right)}{\left(2+\frac{3}{n^2}\right)} \to \frac{3}{2}.$$

We conclude that  $b_n \to \ln\left(\frac{3}{2}\right)$ .

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## Using L'Hôpital's rule on sequences

Theorem (L'Hôpital's rule for sequences)

If the sequence  $\{a_n\}$  satisfies that:

- There exist a function f such that for n > N the sequence elements a<sub>n</sub> can be written as a<sub>n</sub> = f(n);
- And  $\lim_{x\to\infty} f(x) = L;$

then holds that  $\lim_{n\to\infty} a_n = L$ .

Remark: The  $\lim_{x\to\infty} f(x)$  may indeterminate, and L'Hôpital's rule might be used to compute that limit.

Example Find the limit  $a_n = \sqrt[8n]{5n}$  as  $n \to \infty$ .

Solution: Notice that  $a_n = f(n)$  for  $f(x) = \sqrt[8x]{5x}$ .

# Using L'Hôpital's rule on sequences Example Find the limit $a_n = \sqrt[8n]{5n}$ as $n \to \infty$ . Solution: Recall: $a_n = f(n)$ for $f(x) = \sqrt[8x]{5x}$ . $\sqrt[8n]{5x} = e^{\ln(\sqrt[8n]{5x})} = e^{\left(\frac{\ln(5x)}{8x}\right)}$ But $\lim_{x\to\infty} \frac{\ln(5x)}{8x}$ is indeterminate $\frac{\infty}{\infty}$ . L'Hôpital's rule, $\lim_{x\to\infty} \frac{\ln(5x)}{8x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{8} = \lim_{x\to\infty} \frac{1}{8x} = 0.$ $\lim_{x\to\infty} \sqrt[8n]{5x} = \lim_{x\to\infty} e^{\left(\frac{\ln(5x)}{8x}\right)} = e^0 \Rightarrow \lim_{x\to\infty} \sqrt[8n]{5x} = 1.$ We conclude that $\sqrt[8n]{5n} \to 1$ as $n \to \infty$ .

# Using L'Hôpital's rule on sequences

### Example

Given positive numbers *a*, *b*, find the  $\lim_{n\to\infty} \left(1-\frac{b}{n}\right)^{an}$ .

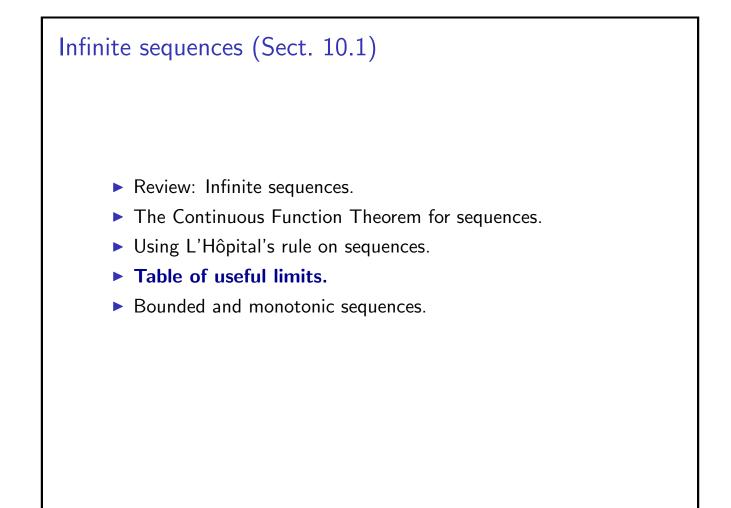
Solution: We rewrite the sequence as follows,

$$\left(1-rac{b}{n}
ight)^{an}=e^{\left[an\ln\left(1-rac{b}{n}
ight)
ight]}=e^{\left[rac{a\ln\left(1-rac{b}{n}
ight)}{rac{1}{n}}
ight]}$$

The exponent has an indeterminate limit,  $\frac{a \ln(1 - \frac{b}{n})}{\frac{1}{n}} \rightarrow \frac{0}{0}$ . Recall the argument with the L'Hôpital's rule on functions,

$$\lim_{x \to \infty} \frac{a \ln(1 - \frac{b}{x})}{\frac{1}{x}} = \lim_{x \to \infty} \frac{\frac{a}{(1 - \frac{b}{x})} \frac{b}{x^2}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{ab}{(1 - \frac{b}{x})} = ab$$

We conclude that  $\lim_{n\to\infty} \left(1-\frac{b}{n}\right)^{an} = e^{ab}$ .



# Table of useful limits

Remark: The following limits appear often in applications:

$$\lim_{n\to\infty}\frac{\ln(n)}{n}=0;$$

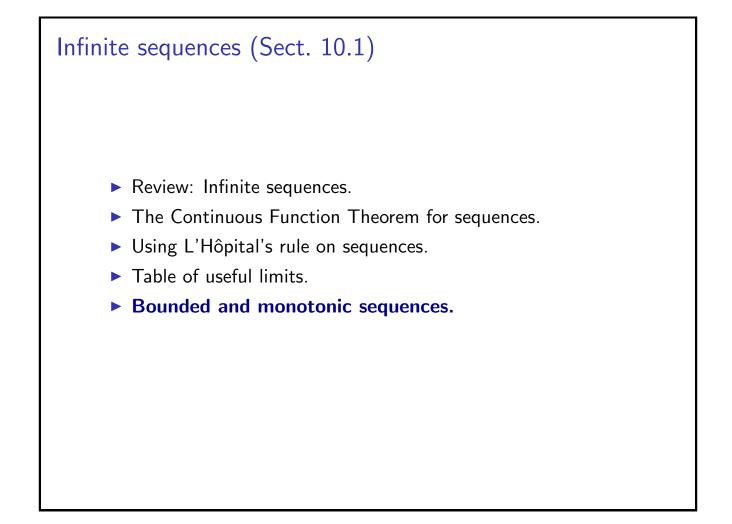
$$\lim_{n\to\infty} \sqrt[n]{n} = 1;$$

$$\lim_{n \to \infty} x^{\left(\frac{1}{n}\right)} = 1, \text{ for } x > 0;$$

• 
$$\lim_{n \to \infty} x^n = 0$$
, for  $|x| < 1$ ;

$$\lim_{n \to \infty} \left( 1 + \frac{x}{n} \right)^n = e^x, \text{ for } x \in \mathbb{R};$$

$$\lim_{n\to\infty}\frac{x^n}{n!}=0.$$



### Bounded and monotonic sequences

### Definition

A sequence  $\{a_n\}$  is bounded above iff there is  $M \in \mathbb{R}$  such that

$$a_n \leqslant M$$
 for all  $n \ge 1$ .

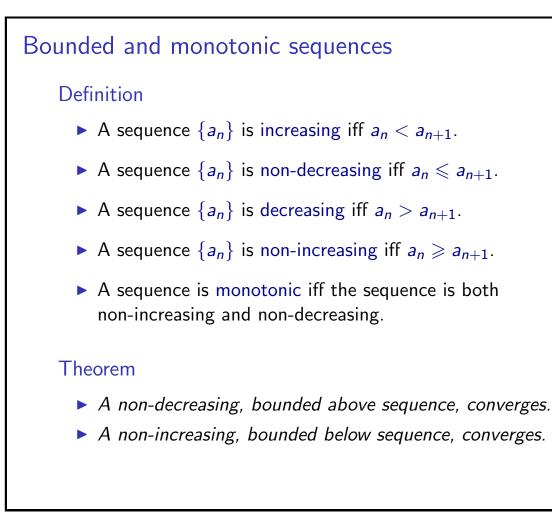
The sequence  $\{a_n\}$  is bounded below iff there is  $m \in \mathbb{R}$  such that

$$m \leqslant a_n$$
 for all  $n \geqslant 1$ .

A sequence is bounded iff it is bounded above and below.

### Example

•  $a_n = \frac{1}{n}$  is bounded, since  $0 < \frac{1}{n} \le 1$ . •  $a_n = (-1)^n$  is bounded, since  $-1 \le (-1)^n \le 1$ .



## Bounded and monotonic sequences

### Example

Determine whether the sequence  $a_n = \frac{n}{n^2 + 1}$  converges or not.

Solution: We show that  $a_n$  is decreasing. Indeed, the condition

$$egin{aligned} a_{n+1} < a_n & \Leftrightarrow & rac{n+1}{(n+1)^2+1} < rac{n}{n^2+1} \ & (n+1)(n^2+1) < n(n^2+2n+2) \ & n^3+n^2+n+1 < n^3+2n^2+2n \end{aligned}$$

Since  $1 < (n^2 + n)$  is true for  $n \ge 1$ , then  $a_{n+1} < a_n$ ; decreasing. The sequence satisfies that  $0 < a_n$ , bounded below.

We conclude that  $a_n$  converges.

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