Infinite sequences (Sect. 10.1)

Today's Lecture:

- Overview: Sequences, series, and calculus.
- Definition and geometrical representations.
- The limit of a sequence, convergence, divergence.
- Properties of sequence limits.
- The Sandwich Theorem for sequences.

Next Lecture:

- The Continuous Function Theorem for sequences.
- Using L'Hôpital's rule on sequences.
- Table of useful limits.
- Bounded and monotonic sequences.

Overview: Sequences, series, and calculus

Remarks:

- We have defined the $\int_a^b f(x)\,dx$ as a limit of partial sums. That is, as an infinite sum of numbers (areas of rectangles).
- In the next section we define, precisely, what is an infinite sum. Infinite sums are called *series*.
- In this section we introduce the idea of an *infinite sequence* of numbers. We will use sequences to define series.
- Later on, the idea of infinite sums will be generalized from numbers to functions.
- We will express differentiable functions as infinite sums of polynomials (Taylor series expansions).
- Then we will be able to compute integrals like $\int_a^b e^{-x^2}\,dx$.

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Definition and geometrical representations

Definition

An infinite sequence of numbers is an ordered set of real numbers.

Remark: A sequence is denoted as

$$\{a_1, a_2, a_3, \cdots, a_n, \cdots\}, \quad \text{or} \quad \{a_n\}_{n=1}^{\infty}, \quad \text{or} \quad \{a_n\}.$$

Example

$$\Big\{\frac{n}{n+1}\Big\}_{n=1}^{\infty}, \quad a_n = \frac{n}{n+1}, \quad \Big\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots, \frac{n}{n+1}, \cdots\Big\}.$$

$$\big\{(-1)^n\sqrt{n}\big\}_{n=3}^{\infty}, \quad a_n = (-1)^n\sqrt{n}, \quad \big\{-\sqrt{3}, \sqrt{4}, -\sqrt{5}, \cdots\big\}.$$

$$\big\{\cos(n\pi/6)\big\}_{n=0}^{\infty}, \quad a_n = \cos(n\pi/6), \quad \Big\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \cdots\Big\}.$$

Definition and geometrical representations

Example

Find a formula for the general term of the sequence

$$\left\{\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \cdots\right\}.$$

Solution: We know that:

$$a_1 = \frac{3}{5}, \quad a_2 = -\frac{4}{25}, \quad a_3 = \frac{5}{125}, \quad a_4 = -\frac{6}{625}.$$

$$a_1 = \frac{(1+2)}{5}, \quad a_2 = -\frac{(2+2)}{5^2}, \quad a_3 = \frac{(3+2)}{5^3}, \quad a_4 = -\frac{(4+2)}{5^4}.$$

We conclude that $a_n = (-1)^{(n-1)} \dfrac{(n+2)}{5^n}$. ◁

Definition and geometrical representations

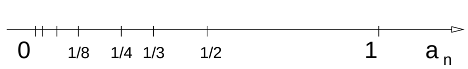
Remark:

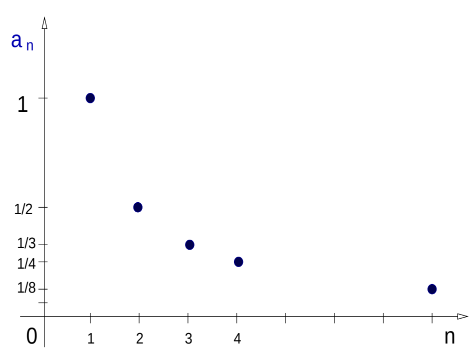
Infinite sequences can be represented on a line or on a plane.

Example

Graph the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ on a line and on a plane.

Solution:





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The limit of a sequence, convergence, divergence

Remark:

- As it happened in the example above, the numbers a_n in a sequence may approach a single value as n increases.

$$\left\{a_n = \frac{1}{n}\right\}_{n=1}^{\infty} = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\right\} \to 0.$$

- This is not the case for every sequence. The sequence elements may grow unbounded:

$$\{n^2\}_{n=1}^{\infty} = \{1, 4, 9, 16, \cdots\}.$$

The sequence numbers may oscillate:

$$\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, \cdots\}.$$

The limit of a sequence, convergence, divergence

Definition

An infinite sequence $\{a_n\}$ has limit L iff for every number $\epsilon > 0$ there exists a positive integer N such that

$$N < n \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

A sequence is called convergent iff it has a limit, otherwise it is called divergent.

Remark: We use the notation $\lim_{n\to\infty} a_n = L$ or $a_n \to L$.

Example

Find the limit of the sequence $\left\{a_n = 1 + \dfrac{3}{n^2}\right\}_{n=1}^{\infty}$.

Solution: Since $\dfrac{1}{n^2} \to 0$, we will prove that $L = 1$.

The limit of a sequence, convergence, divergence

Example

Find the limit of the sequence $\left\{a_n = 1 + \dfrac{3}{n^2}\right\}_{n=1}^{\infty}$.

Solution: Recall: The candidate for limit is $L = 1$.

Given any $\epsilon > 0$, we need to find the appropriate N. Since

$$|a_n - 1| < \epsilon \quad \Leftrightarrow \quad \left|\frac{3}{n^2}\right| < \epsilon \quad \Leftrightarrow \quad \frac{3}{\epsilon} < n^2 \quad \Leftrightarrow \quad \sqrt{\frac{3}{\epsilon}} < n.$$

Therefore, given $\epsilon > 0$, choose $N = \sqrt{\dfrac{3}{\epsilon}}$.

We then conclude that for all $n > N$ holds,

$$\sqrt{\frac{3}{\epsilon}} < n \quad \Leftrightarrow \quad \frac{3}{\epsilon} < n^2 \quad \Leftrightarrow \quad \left|\frac{3}{n^2}\right| < \epsilon \quad \Leftrightarrow \quad |a_n - 1| < \epsilon.$$

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Properties of sequence limits

Remark: The limits of simple sequences can be used to compute limits of more complicated sequences.

Theorem (Limit properties)

If the sequence $\{a_n\} \to A$ *and* $\{b_n\} \to B$*, then holds,*

(a) $\lim_{n\to\infty} \{a_n + b_n\} = A + B;$

(b) $\lim_{n\to\infty} \{a_n - b_n\} = A - B;$

(c) $\lim_{n\to\infty} \{ka_n\} = kA;$

(d) $\lim_{n\to\infty} \{a_n b_n\} = AB;$

(e) *If* $B \neq 0$*, then* $\lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}.$

Properties of sequence limits

Example

Find the limit of the sequence $\left\{a_n = \dfrac{1-2n}{2+3n}\right\}_{n=1}^{\infty}$.

Solution: We use the properties above to find the limit.

Rewrite the sequence as follows,

$$a_n = \frac{(1-2n)}{(2+3n)} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \frac{\frac{1}{n}-2}{\frac{2}{n}+3}.$$

Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, then

$$\frac{1}{n} - 2 \to -2, \qquad \frac{2}{n} \to 0, \qquad \frac{2}{n} + 3 \to 3.$$

Hence, the quotient property implies $a_n \to -\dfrac{2}{3}$. ◁

Properties of sequence limits

Example

Find the limit of the sequence $\left\{a_n = \dfrac{3n^3-2n+1}{2n^2+4}\right\}_{n=1}^{\infty}$.

Solution: Rewrite the sequence as follows,

$$a_n = \frac{(3n^3-2n+1)}{(2n^2+4)} \frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)} = \frac{3n - \frac{2}{n} + \frac{1}{n^2}}{2 + \frac{4}{n^2}}$$

Since $\dfrac{1}{n} \to 0$ as $n \to \infty$, then

$$\frac{1}{n^2} = \left(\frac{1}{n}\right)^2 \to 0, \qquad \frac{2}{n} \to 0, \qquad 2 + \frac{4}{n^2} \to 2.$$

Hence, the quotient property implies $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{3n}{2}$.

We conclude that a_n diverges. ◁

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The Sandwich Theorem for sequences

Theorem (Sandwich-Squeeze)

If the sequences $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ satisfy

$$a_n \leqslant b_n \leqslant c_n, \quad \text{for} \quad n > N,$$

and if both $a_n \to L$ and $c_n \to L$, then holds

$$b_n \to L.$$

Example

Find the limit of the sequence $\left\{ a_n = \dfrac{\sin(3n)}{n^2} \right\}_{n=1}^{\infty}$.

Solution: Since $|\sin(3n)| \leqslant 1$, then

$$|a_n| = \left|\frac{\sin(3n)}{n^2}\right| \leqslant \left|\frac{1}{n^2}\right| = \frac{1}{n^2} \quad \Rightarrow \quad -\frac{1}{n^2} \leqslant a_n \leqslant \frac{1}{n^2}.$$

Since $\pm\dfrac{1}{n^2} \to 0$, we conclude that $a_n \to 0$. $\triangleleft$