Improper integrals (Sect. 8.7)

- Review: Improper integrals type I and II.
- Examples: $I = \int_1^{\infty} \frac{dx}{x^p}$, and $I = \int_0^1 \frac{dx}{x^p}$.
- Convergence test: Direct comparison test.
- Convergence test: Limit comparison test.

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Review: Improper integrals type I

Definition (Type I)

Improper integrals of Type I are integrals of continuous functions on infinite domains; these include:

The improper integral of a continuous function f on $[a, \infty)$,

$$\int_a^\infty f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function f on $(-\infty, b]$,

$$\int_{-\infty}^b f(x)\,dx = \lim_{a \to -\infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function f on $(-\infty, \infty)$,

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx.$$

Review: Improper integrals type II

Definition (Type II)

Improper integrals of Type II are integrals of functions with vertical asymptotes within the integration interval; these include:

If f is continuous on $(a, b]$ and discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

If f is continuous on $[a, b)$ and discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

If f is continuous on $[a, c) \cup (c, b]$ and discontinuous at c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

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- **Examples:** $I = \int_1^{\infty} \frac{dx}{x^p}$, **and** $I = \int_0^1 \frac{dx}{x^p}$.
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The cases $\int_0^1 \frac{dx}{x^p}$ and $\int_1^{\infty} \frac{dx}{x^p}$

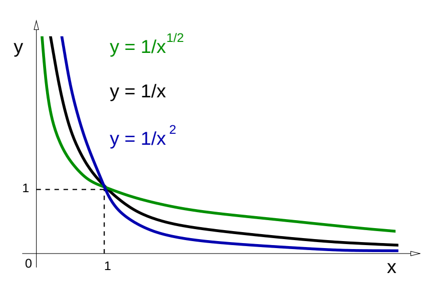
Summary: In the case $p = 1$ both integrals diverge,

$$\int_0^1 \frac{dx}{x} = \text{diverges}, \qquad \int_1^{\infty} \frac{dx}{x} = \text{diverges}.$$

In the case $p \neq 1$ we have:

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} = \dfrac{1}{1-p} & p < 1, \\ \text{diverges} & p > 1. \end{cases}$$

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \text{diverges} & p < 1, \\ = \dfrac{1}{p-1} & p > 1. \end{cases}$$



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- Examples: $I = \int_1^{\infty} \frac{dx}{x^p}$, and $I = \int_0^1 \frac{dx}{x^p}$.
- **Convergence test: Direct comparison test.**
- Convergence test: Limit comparison test.

Convergence test: Direct comparison test

Remark: Convergence tests determine whether an improper integral converges or diverges.

Theorem (Direct comparison test)

If functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and $0 \leqslant f(x) \leqslant g(x)$ for every $x \in [a, \infty)$, then holds

$$0 \leqslant \int_a^{\infty} f(x)\,dx \leqslant \int_a^{\infty} g(x)\,dx.$$

The inequalities above imply the following statements:

(a) $\int_a^{\infty} g(x)\,dx$ *converges* $\Rightarrow$ $\int_a^{\infty} f(x)\,dx$ *converges;*

(b) $\int_a^{\infty} f(x)\,dx$ *diverges* $\Rightarrow$ $\int_a^{\infty} g(x)\,dx$ *diverges.*

Convergence test: Direct comparison test

Example

Determine whether $I = \int_1^{\infty} e^{-x^2}\, dx$ converges or diverges.

Solution: Notice that $\int e^{-x^2}\, dx$ does not have an expression in terms of elementary functions. However,

$$1 \leqslant x \quad \Rightarrow \quad x \leqslant x^2 \quad \Rightarrow \quad -x^2 \leqslant -x \quad \Rightarrow \quad e^{-x^2} \leqslant e^{-x}.$$

The last inequality follows because exp is an increasing function.

$$0 \leqslant \int_1^{\infty} e^{-x^2}\, dx \leqslant \int_1^{\infty} e^{-x}\, dx = -e^{-x}\Big|_1^{\infty} = \frac{1}{e}.$$

Since $0 \leqslant \int_1^{\infty} e^{-x^2}\, dx \leqslant \frac{1}{e}$, the integral converges. ◁

Convergence test: Direct comparison test

Example

Determine whether $I = \int_1^{\infty} \frac{dx}{\sqrt{x^6+1}}$ converges or diverges.

Solution: We need to find an appropriate function to compare with the integrand above. We need to find either

- a bigger function with convergent integral;
- or a smaller function with divergent integral.

Notice: $x^6 < x^6+1 \quad \Rightarrow \quad x^3 < \sqrt{x^6+1} \quad \Rightarrow \quad \frac{1}{\sqrt{x^6+1}} < \frac{1}{x^3}.$

Therefore, $0 < \int_1^{\infty} \frac{dx}{\sqrt{x^6+1}} < \int_1^{\infty} \frac{dx}{x^3} = -\frac{x^{-2}}{2}\Big|_1^{\infty} = \frac{1}{2}.$

Since $0 \leqslant \int_1^{\infty} \frac{dx}{\sqrt{x^6+1}} \leqslant \frac{1}{2}$, the integral converges. ◁

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- **Convergence test: Limit comparison test.**

Convergence test: Limit comparison test

Remark: Convergence tests determine whether an improper integral converges or diverges.

Theorem (Limit comparison test)

If positive functions $f, g : [a, \infty) \to \mathbb{R}$ are continuous and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad \text{with} \quad 0 < L < \infty,$$

then the integrals

$$\int_a^{\infty} f(x)\, dx, \qquad \int_a^{\infty} g(x)\, dx$$

both converge or both diverge.

Remark: Although both integrals above may converge, their values need **not** be the same.

Convergence test: Limit comparison test

Example

Determine whether $I = \int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges or diverges.

Solution: The convergence of integrals involving rational functions is simple to determine with the limit comparison test.

First, determine the behavior of the rational function as $x \to \infty$;

$$\frac{1}{\sqrt{x^6+1}} \to \frac{1}{x^3}, \quad \text{as} \quad x \to \infty.$$

Then, chose the limit comparison function $g(x) = 1/x^3$; since

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/\sqrt{x^6+1}}{1/x^3} = \lim_{x\to\infty} \frac{x^3}{\sqrt{x^6+1}} = 1.$$

Since $\int_1^\infty \frac{dx}{x^3}$ converges, then $\int_1^\infty \frac{dx}{\sqrt{x^6+1}}$ converges too. ◁

Convergence test: Limit comparison test

Example

Determine whether $I = \int_3^\infty \frac{dx}{(2+\cos(x)+\ln(x))}$ converges or not.

Solution: Choose the comparison function $g(x) = 1/\ln(x)$;

$$\lim_{x\to\infty} \frac{1/(2+\cos(x)+\ln(x))}{1/\ln(x)} = \lim_{x\to\infty} \frac{\ln(x)}{(2+\cos(x)+\ln(x))} = 1.$$

The limit comparison test says: The integral I converges iff $J = \int_3^\infty \frac{dx}{\ln(x)}$ converges. We need to find out if J converges.

We now use the direct comparison test: for $x > 0$ holds

$$\ln(x) < x \quad \Rightarrow \quad \frac{1}{x} < \frac{1}{\ln(x)} \quad \Rightarrow \quad \int_3^\infty \frac{dx}{x} < \int_3^\infty \frac{dx}{\ln(x)}.$$

Since $\int_3^\infty \frac{dx}{x}$ diverges, then both J and I diverge. ◁

Convergence test: Limit comparison test

Example

Determine whether $I = \int_3^{\infty} \frac{x\,dx}{\sqrt{x^5 + x^3}}$ converges or not.

Solution: First, find an appropriate function $g(x)$ such that:

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \frac{x}{\sqrt{x^5 + x^3}} = \lim_{x\to\infty} \frac{x}{x^{5/2}} = \lim_{x\to\infty} \frac{1}{x^{3/2}}.$$

Therefore, we use the limit comparison test with $g(x) = x^{-3/2}$. Then, by construction,

$$\lim_{x\to\infty} \Big(\frac{x}{\sqrt{x^5 + x^3}}\Big)\Big(\frac{1}{x^{-3/2}}\Big) = \lim_{x\to\infty} \Big(\frac{x}{x^{5/2}}\Big)\Big(\frac{1}{x^{-3/2}}\Big) = 1.$$

Since $\int_3^{\infty} x^{-3/2}\,dx = -2\,x^{-1/2}\Big|_3^{\infty} = -2\Big(0 - \frac{1}{\sqrt{3}}\Big) = \frac{2}{\sqrt{3}}$,

we conclude that I converges. ◁