## Improper integrals (Sect. 8.7)

This class:

- Integrals on infinite domains (Type I).
- The case $I = \displaystyle\int_1^\infty \frac{dx}{x^p}$.
- Integrands with vertical asymptotes (Type II).
- The case $I = \displaystyle\int_0^1 \frac{dx}{x^p}$.

Next class:

- Convergence tests:
  - Direct comparison test.
  - Limit comparison test.
- Examples.

## Improper integrals (Sect. 8.7)

- **Integrals on infinite domains (Type I).**
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## Integrals on infinite domains (Type I)

Remark: Improper integrals are the limit of definite integrals when one endpoint if integration approaches $\pm\infty$.

### Definition (Type I)

Improper integrals of Type I are integrals of continuous functions on infinite domains; these include:

The improper integral of a continuous function $f$ on $[a, \infty)$,

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function $f$ on $(-\infty, b]$,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

The improper integral of a continuous function $f$ on $(-\infty, \infty)$,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx.$$

## Integrals on infinite domains (Type I)

Example

Evaluate $I = \displaystyle\int_{-\infty}^{-1} \frac{\ln(|x|)}{x^2}\,dx.$

Solution: This is an improper integral:

$$\int_{-\infty}^{-1} \frac{\ln(|x|)}{x^2}\,dx = \lim_{a\to-\infty} \int_a^{-1} \frac{\ln(|x|)}{x^2}$$

Integrating by parts, $u = \ln(|x|)$, and $dv = dx/x^2$,

$$\int_a^{-1} \frac{\ln(|x|)}{x^2}\,dx = \left(-\frac{1}{x}\right) \ln(|x|)\Big|_a^{-1} - \int_a^{-1} \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right)dx$$

$$\int_a^{-1} \frac{\ln(|x|)}{x^2}\,dx = \frac{\ln(|a|)}{a} + \int_a^{-1} \frac{dx}{x^2} = \frac{\ln(|a|)}{a} - \frac{1}{x}\Big|_a^{-1}.$$

## Integrals on infinite domains (Type I)

**Example**

Evaluate $I = \int_{-\infty}^{-1} \frac{\ln(|x|)}{x^2}\,dx.$

Solution: Recall: $\int_{-\infty}^{-1} \frac{\ln(|x|)}{x^2}\,dx = \lim_{a\to-\infty}\Big(\frac{\ln(|a|)}{a} + 1 + \frac{1}{a}\Big).$

The first limit on the right-hand side is indeterminate $-\frac{\infty}{\infty}$.

L'Hôpital's rule implies

$$\lim_{a\to-\infty}\frac{\ln(|a|)}{a} = \lim_{a\to-\infty}\frac{(\ln(|a|))'}{(a)'} = \lim_{a\to-\infty}\frac{(1/a)}{1} = 0.$$

Therefore, the improper integral is given by

$$\int_{-\infty}^{-1}\frac{\ln(|x|)}{x^2}\,dx = 0 + 1 + 0 \quad\Rightarrow\quad \int_{-\infty}^{-1}\frac{\ln(|x|)}{x^2}\,dx = 1. \qquad \triangleleft$$

## Integrals on infinite domains (Type I)

**Example**

Evaluate $I = \int_{3}^{\infty} \frac{2}{x^2-2x}\,dx.$

Solution: Recall: $I = \lim_{b\to\infty}\int_3^b \frac{2}{x^2-2x}\,dx.$

First integrate, then the limit. For the integral, partial fractions:

$$\frac{2}{x^2-2x} = \frac{2}{x(x-2)} = \frac{a}{x} + \frac{b}{(x-2)} \quad\Rightarrow\quad 2 = a(x-2) + bx.$$

Then $a = -1$, and $b = 1$. Hence

$$\int_3^b \frac{2}{x^2-2x}\,dx = -\int_3^b \frac{dx}{x} + \int_3^b \frac{dx}{x-2} = \big[-\ln x + \ln(x-2)\big]\Big|_3^b.$$

$$\int_3^b \frac{2}{x^2-2x}\,dx = \ln(1/b) + \ln(b-2) + \ln(3).$$

## Integrals on infinite domains (Type I)

**Example**

Evaluate $I = \int_3^{\infty} \frac{2}{x^2 - 2x}\, dx.$

Solution: Recall: $\int_3^{b} \frac{2}{x^2 - 2x}\, dx = \ln(1/b) + \ln(b-2) + \ln(3).$

Therefore, the improper integral is

$$I = \lim_{b\to\infty} \Big[ \ln\Big(\frac{b-2}{b}\Big) + \ln(3) \Big].$$

The natural log function is continuous,

$$I = \ln\Big( \lim_{b\to\infty} \frac{b-2}{b} \Big) + \ln(3) = \ln(1) + \ln(3).$$

We then conclude that $I = \ln(3)$. $\triangleleft$

## Improper integrals (Sect. 8.7)

- Integrals on infinite domains (Type I).
- **The case $I = \int_1^{\infty} \frac{dx}{x^p}$.**
- Integrands with vertical asymptotes (Type II).
- The case $I = \int_0^{1} \frac{dx}{x^p}$.

## The case $I = \int_1^\infty \frac{dx}{x^p}$

Example

Evaluate $I = \int_1^\infty \frac{dx}{x^p}$ for $p > 0$.

Solution: We first compute the integral, then take the limit.

For $p \neq 1$, holds $I = \lim_{b\to\infty} \int_1^b x^{-p}\,dx = \lim_{b\to\infty} \Big[ \frac{x^{(-p+1)}}{(-p+1)}\Big|_1^b \Big]$.

$$I = \frac{(-1)}{(p-1)}\Big[\lim_{b\to\infty} \frac{1}{b^{(p-1)}} - 1\Big] \quad \Rightarrow \quad \begin{cases} I \text{ diverges} & p < 1, \\ I = \dfrac{1}{p-1} & p > 1. \end{cases}$$

In the case $p = 1$ the integral diverges since $I = \lim_{b\to\infty} \ln(b)$. $\triangleleft$

## Improper integrals (Sect. 8.7)

- Integrals on infinite domains (Type I).
- The case $I = \int_1^\infty \frac{dx}{x^p}$.
- **Integrands with vertical asymptotes (Type II).**
- The case $I = \int_0^1 \frac{dx}{x^p}$.

## Integrands with vertical asymptotes (Type II)

### Definition (Type II)

Improper integrals of Type II are integrals of functions with vertical asymptotes within the integration interval; these include:

If $f$ is continuous on $(a, b]$ and discontinuous at $a$, then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

If $f$ is continuous on $[a, b)$ and discontinuous at $b$, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

If $f$ is continuous on $[a, c) \cup (c, b]$ and discontinuous at $c$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

## Integrands with vertical asymptotes (Type II)

### Example

Evaluate $I = \displaystyle\int_0^1 \frac{dx}{(1-x)^2}$.

Solution: Recall: $I = \displaystyle\lim_{c \to 1^-} \int_0^c (1-x)^{-2}\,dx$.

We first integrate, then we take the limit. The integral is

$$\int_0^c (1-x)^{-2}\,dx = (1-x)^{-1}\Big|_0^c = \frac{1}{(1-c)} - 1.$$

We now take the limit,

$$I = \lim_{c \to 1^-} \frac{1}{(1-c)} - 1.$$

We conclude that $I$ diverges. $\triangleleft$

## Integrands with vertical asymptotes (Type II)

Example

Evaluate $I = \int_0^2 \frac{dx}{(1-x)^{2/5}}$.

Solution:

Recall: $\int_0^2 \frac{dx}{(1-x)^{2/5}} = \int_0^1 \frac{dx}{(1-x)^{2/5}} + \int_1^2 \frac{dx}{(1-x)^{2/5}}$.

The first integral is given by

$$\int_0^1 \frac{dx}{(1-x)^{2/5}} = \lim_{c\to 1^-} \int_0^c (1-x)^{-2/5}\,dx = \lim_{c\to 1^-} -\frac{5}{3}\,(1-x)^{3/5}\Big|_0^c,$$

$$\int_0^1 \frac{dx}{(1-x)^{2/5}} = -\frac{5}{3}\lim_{c\to 1^-}(1-c)^{3/5} + \frac{5}{3} = \frac{5}{3}.$$

## Integrands with vertical asymptotes (Type II)

Example

Evaluate $I = \int_0^2 \frac{dx}{(1-x)^{2/5}}$.

Solution: Recall: $\int_0^2 \frac{dx}{(1-x)^{2/5}} = \frac{5}{3} + \int_1^2 \frac{dx}{(1-x)^{2/5}}$.

The second integral is given by

$$\int_1^2 \frac{dx}{(1-x)^{2/5}} = \lim_{c\to 1^+} \int_c^2 (1-x)^{-2/5}\,dx = \lim_{c\to 1^+} -\frac{5}{3}\,(1-x)^{3/5}\Big|_c^2,$$

$$\int_1^2 \frac{dx}{(1-x)^{2/5}} = \frac{5}{3} + \frac{5}{3}\lim_{c\to 1^+}(1-c)^{3/5} = \frac{5}{3}.$$

We conclude: $I = \frac{10}{3}$. ◁

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- **The case $I = \displaystyle\int_0^{1} \frac{dx}{x^p}$.**

## The case $I = \displaystyle\int_0^{1} \frac{dx}{x^p}$

Example

Evaluate $I = \displaystyle\int_0^{1} \frac{dx}{x^p}$ for $p > 0$.

Solution: We first compute the integral, then take the limit.

For $p \neq 1$, holds $I = \displaystyle\lim_{a\to 0^+} \int_a^1 x^{-p}\,dx = \lim_{a\to 0^+} \Big[\frac{x^{(-p+1)}}{(-p+1)}\Big|_a^1\Big].$

$$I = \frac{(-1)}{(p-1)}\Big[1 - \lim_{a\to 0^+} \frac{1}{a^{(p-1)}}\Big] \quad \Rightarrow \quad \begin{cases} I = \dfrac{1}{1-p} & p < 1, \\ I \text{ diverges} & p > 1. \end{cases}$$

In the case $p = 1$ the integral diverges since $I = \displaystyle\lim_{a\to 0^+} \ln(a)$. $\lhd$

## The cases $\int_0^1 \frac{dx}{x^p}$ and $\int_1^\infty \frac{dx}{x^p}$

Summary:

$$\int_0^1 \frac{dx}{x^p} = \begin{cases} = \dfrac{1}{1-p} & p < 1, \\ \text{diverges} & p > 1. \end{cases}$$

$$\int_1^\infty \frac{dx}{x^p} = \begin{cases} \text{diverges} & p < 1, \\ = \dfrac{1}{p-1} & p > 1. \end{cases}$$

In the case $p = 1$ both integrals diverge,

$$\int_0^1 \frac{dx}{x} = \text{diverges}, \qquad \int_1^\infty \frac{dx}{x} = \text{diverges}.$$