Integrating using tables (Sect. 8.5)

- Remarks on:
  - Using Integration tables.
  - Reduction formulas.
  - Computer Algebra Systems.
  - Non-elementary integrals.
- Limits using L'Hôpital's Rule (Sect. 7.5).

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Using Integration tables

Remark: Sometimes to use integration tables one needs to rewrite the integral in the form that appears in the table.

Example

Evaluate $I = \int \frac{dx}{\sqrt{2x^3 + 3x^2}}$, for $x > 0$.

Solution: We start rewriting our integral as

$$I = \int \frac{dx}{\sqrt{x^2(2x+3)}} = \int \frac{dx}{|x|\,\sqrt{2x+3}} = \int \frac{dx}{x\,\sqrt{2x+3}},$$

where we used that $x > 0$. Notice that the denominator does not vanishes for $x > 0$. After looking for a while in the integration tables at the end of the textbook, we find the entry (13b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right| + c.$$

Using Integration tables

Example

Evaluate $I = \int \frac{dx}{\sqrt{2x^3 + 3x^2}}$, for $x > 0$.

Solution: Recall: $I = \int \frac{dx}{x\,\sqrt{2x+3}}$ and from the table,

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}}\right| + c.$$

We can use this formula for $a = 2$ and $b = 3$. We conclude that

$$\int \frac{dx}{x\sqrt{2x+3}} = \frac{1}{\sqrt{3}} \ln\left|\frac{\sqrt{2x+3} - \sqrt{3}}{\sqrt{2x+3} + \sqrt{3}}\right| + c. \qquad \lhd$$

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Reduction formulas

Remark: Sometimes integration tables only relates two integrals.

Example

Evaluate $I = \displaystyle\int \frac{dx}{\sqrt{4x^5 + 9x^4}}$, for $x > 0$.

Solution: We can rewrite the integral as

$$I = \int \frac{dx}{\sqrt{x^4(4x+9)}} = \int \frac{dx}{x^2\sqrt{(4x+9)}}.$$

Entry (15) in the integration tables at the end of the textbook is

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b}\int \frac{dx}{x\sqrt{ax+b}}.$$

This formula relates a complicated integral to a simpler integral.

$$\int \frac{dx}{x^2\sqrt{(4x+9)}} = -\frac{\sqrt{4x+9}}{9x} - \frac{2}{9}\int \frac{dx}{x\sqrt{4x+9}}.$$

Reduction formulas

Example

Evaluate $I = \int \frac{dx}{\sqrt{4x^5 + 9x^4}}$, for $x > 0$.

Solution:

Recall: $\displaystyle\int \frac{dx}{x^2\sqrt{(4x+9)}} = -\frac{\sqrt{4x+9}}{9x} - \frac{2}{9}\int \frac{dx}{x\sqrt{4x+9}}$.

We now use the entry (13b) again,

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln\left|\frac{\sqrt{ax+b}-\sqrt{b}}{\sqrt{ax+b}+\sqrt{b}}\right| + c,$$

and we get

$$I = -\frac{\sqrt{4x+9}}{9x} - \frac{2}{9}\left[\frac{1}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right|\right] + c. \qquad \triangleleft$$

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Computer Algebra Systems

Remarks:

- Programs like Mathematica and Maple can be used to compute analytic expression for integrals.
- Different programs can provide equivalent, but not identical, expressions for the same integral.

Example

Use Maple and Mathematica to evaluate $I = \int x^2 \sqrt{a^2 + x^2}\, dx$.

Solution: Maple gives:

$$I = \frac{x}{4}(a^2 + x^2)^{3/2} - \frac{a^2 x}{8}\sqrt{a^2 + x^2} - \frac{a^2}{8}\ln(x + \sqrt{a^2 + x^2}).$$

Mathematica gives

$$\Big(\frac{a^2 x}{8} + \frac{x^3}{4}\Big)\sqrt{a^2 + x^2} - \frac{a^2}{8}\ln(x + \sqrt{a^2 + x^2}).$$

Both expressions define the same function. $\triangleleft$

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Non-elementary integrals

Remarks:

- Integration is more difficult that derivation.
- The derivative of an elementary function is again an elementary function.
- Elementary functions: polynomials, rational powers of quotient of polynomials, trigonometric functions.
- A similar statement is not true for integration.
- Example: $f(x) = \int \frac{dx}{x}$ is a new function. It is called $\ln(x)$.
- In a similar way, the following integrals define new functions:

$$\mathrm{erf} = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt, \quad I_1 = \int \sin(x^2)\, dx, \quad I_2 = \int \frac{\sin(x)}{x}\, dx$$

$$I_2 = \int \sqrt{1 + x^4}\, dx, \quad I_3 = \int \frac{e^x}{x}\, dx, \quad I_4 = \int \frac{dx}{\ln(x)}.$$

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Limits using L'Hôpital's Rule (Sect. 7.5)

Remarks:

- L'Hôpital's rule applies on limits of the form $L = \lim_{x \to a} \frac{f(x)}{g(x)}$ in the case that $f(a) = 0$ and $g(a) = 0$.
- These limits are called indeterminate and denoted as $\frac{0}{0}$.

Theorem

If functions $f, g : I \to \mathbb{R}$ are differentiable in an open interval containing $x = a$, with $f(a) = g(a) = 0$ and $g'(x) \neq 0$ for $x \in I - \{a\}$, then holds

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

assuming the limit on the right-hand side exists.

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

Solution: This limit can be easily computed using L'Hôpital's rule.

The limit is indeterminate, $\frac{0}{0}$. But,

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = 1.$$

We conclude $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$. $\triangleleft$

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $L = \lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$.

Solution: The limit is indeterminate, $\dfrac{0}{0}$. But,

$$L = \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - (1/2)}{2x}.$$

The limit on the right-hand side is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a second time,

$$L = \lim_{x\to 0} \frac{(-1/4)(1+x)^{-3/2}}{2} = \frac{(-1/4)}{2}.$$

We conclude that $L = -\dfrac{1}{8}$. $\triangleleft$

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $L = \lim_{x\to 0} \dfrac{x(1-\cos(6x))}{(7x - \sin(7x))}$.

Solution: The limit is indeterminate, $\dfrac{0}{0}$. But,

$$L = \lim_{x\to 0} \frac{(x - x\cos(6x))}{(7x - \sin(7x))} = \lim_{x\to 0} \frac{1 - \cos(6x) + 6x\sin(6x)}{(7 - 7\cos(7x))}$$

The limit on the right-hand side is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a second time,

$$L = \lim_{x\to 0} \frac{2(6)\sin(6x) + 6^2 x\cos(6x)}{7^2\sin(7x)}$$

Limits using L'Hôpital's Rule (Sect. 7.5)

Example

Evaluate $L = \lim_{x\to 0} \dfrac{x(1-\cos(6x))}{(7x - \sin(7x))}$.

Solution: Recall: $L = \lim_{x\to 0} \dfrac{2(6)\sin(6x) + 6^2 x\cos(6x)}{7^2 \sin(7x)}$.

This limit is still indeterminate, $\dfrac{0}{0}$.

We use L'Hôpital's rule for a third time,

$$L = \lim_{x\to 0} \frac{2(6^2)\cos(6x) + 6^2\cos(6x) + 6^3 x\sin(6x)}{7^3\cos(7x)} = \frac{3(6^2)}{7^3}.$$

We conclude that $L = \dfrac{3(6^2)}{7^3}$. $\triangleleft$