Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- Polynomial division: $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}, \quad k < n.$
- The method of partial fractions.
  - The case $\dfrac{p_1(x)}{(x - r_1)(x - r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
  - The case $\dfrac{p_{(n-1)}(x)}{(x - r_1)^n}$. (Repeated roots).
  - The case $\dfrac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}$, $b^2 - 4c < 0$ (Complex roots).
  - The general case.

Integrating rational functions

Remark:

The problem is to integrate rational functions $f(x) = \dfrac{p_m(x)}{q_n(x)}$, where $p_m(x)$, $q_m(x)$ are polynomials degree m, and n.

Example

Evaluate $I = \displaystyle\int \frac{(5x - 3)}{(x^2 - 2x - 3)}\, dx.$

Solution:

It can be proven that $\dfrac{(5x - 3)}{(x^2 - 2x - 3)} = \dfrac{2}{x + 1} + \dfrac{3}{x - 3}.$

Then, integration is simple: $I = 2 \ln|x + 1| + 3 \ln|x - 3| + c.$ $\triangleleft$

Remark: We now present a method to simplify functions $f(x) = \dfrac{p_m(x)}{q_n(x)}$, as additions of functions simpler to integrate.

Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- **Polynomial division:** $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}, \quad k < n.$
- The method of partial fractions.
  - The case $\dfrac{p_1(x)}{(x-r_1)(x-r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
  - The case $\dfrac{p_{(n-1)}(x)}{(x-r_1)^n}$. (Repeated roots).
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Polynomial division

Remark:
Before start any integration, use long division to simplify the rational function:

$$f(x) = \frac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \frac{r_k(x)}{q_n(x)}, \quad k < n.$$

Remark: Here p_m and q_m are arbitrary polynomials, while r_k is a polynomial with degree less than q_n.

Example
Verify that $\dfrac{4x^2-7}{2x+3} = 2x - 3 + \dfrac{2}{2x+3}$.

Solution:

$$2x - 3 + \frac{2}{2x+3} = \frac{(2x-3)(2x+3)+2}{2x+3} = \frac{4x^2-9+2}{2x+3}. \quad \triangleleft$$

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$\begin{array}{r|l} & 2x - 3 \\ \hline 2x+3 & 4x^2 \qquad - 7 \\ & -4x^2 - 6x \\ \hline & \quad -6x - 7 \\ & \quad 6x + 9 \\ \hline & \qquad 2 \end{array} \qquad \Rightarrow \qquad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.$$

$$I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3} \qquad \Rightarrow \qquad I = x^2 - 3x + \ln(2x + 3) + c.$$

Integrating rational functions (Sect. 8.4)

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- **The method of partial fractions.**
  - The case $\frac{p_1(x)}{(x - r_1)(x - r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
  - The case $\frac{p_{(n-1)}(x)}{(x - r_1)^n}$. (Repeated roots).
  - The case $\frac{p_{(2n-1)}(x)}{(x^2 + bx + c)^n}$, $b^2 - 4c < 0$ (Complex roots).
  - The general case.

The method of partial fractions

Remarks:

- We study rational functions $\dfrac{r_k(x)}{q_n(x)}$, with $k < n$.
- Example: $\dfrac{(5x-3)}{(x+1)(x-3)} = \dfrac{2}{(x+1)} + \dfrac{3}{(x-3)}$.
- The method is called of *partial fractions* because the denominators on the right-hand side above contain only part of the denominator on the left-hand side.
- We present the method through examples.
- We go from simpler to more complicated situations.

Integrating rational functions (Sect. 8.4)

- Integrating rational functions, $\dfrac{p_m(x)}{q_n(x)}$.
- Polynomial division: $\dfrac{p_m(x)}{q_n(x)} = d_{m-n}(x) + \dfrac{r_k(x)}{q_n(x)}$, $k < n$.
- **The method of partial fractions.**
  - **The case** $\dfrac{p_1(x)}{(x-r_1)(x-r_2)}$ $r_1 \neq r_2$ **(Non-repeated roots).**
  - The case $\dfrac{p_{(n-1)}(x)}{(x-r_1)^n}$. (Repeated roots).
  - The case $\dfrac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}$, $b^2 - 4c < 0$ (Complex roots).
  - The general case.

The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{1}{(x-1)(x+2)}\, dx$.

Solution: Denote $r_1 = 1$, $r_2 = -2$. Find a_1 and a_2 such that

$$\frac{1}{(x-1)\,(x+2)} = \frac{a_1}{(x-1)} + \frac{a_2}{(x+2)} = \frac{a(x+2)+b(x-1)}{(x-1)(x+2)}.$$

$$1 = a_1(x+2) + a_2(x-1).$$

To find a_1 evaluate the equation above at the root $r_1 = 1$,

$$1 = a_1(3) \quad \Rightarrow \quad a_1 = \frac{1}{3}.$$

To find a_2 evaluate the equation above at the root $r_2 = -2$,

$$1 = a_2(-3) \quad \Rightarrow \quad a_2 = -\frac{1}{3}.$$

The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{1}{(x-1)(x+2)}\, dx$.

Solution: Recall: $\frac{1}{(x-1)\,(x+2)} = \frac{a_1}{(x-1)} + \frac{a_2}{(x+2)}$,

with $a_1 = \frac{1}{3}$, $a_2 = -\frac{1}{3}$. The integral is now simple to evaluate,

$$I = \int \frac{1}{(x-1)(x-2)}\, dx = \int \frac{1}{3}\,\frac{1}{(x-1)}\, dx - \int \frac{1}{3}\,\frac{1}{(x+2)}\, dx$$

We conclude that

$$I = \frac{1}{3}\,\ln|x-1| - \frac{1}{3}\,\ln|x+2| + c. \qquad \triangleleft$$

The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \displaystyle\int \frac{(x-1)}{(x^2 - x - 2)}\, dx.$

Solution: First, find the zeros of the denominator,

$$x^2 - x - 2 = 0 \quad \Rightarrow \quad x_\pm = \frac{1}{2}\big[1 \pm \sqrt{1+8}\big] \quad \Rightarrow \quad \begin{cases} x_+ = 2, \\ x_- = -1, \end{cases}$$

Therefore, we rewrite: $I = \displaystyle\int \frac{(x-1)}{(x-2)(x+1)}\, dx.$

Partial fraction problem: Find constants a_1 and a_2 such that

$$\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1}{(x-2)} + \frac{a_2}{(x+1)}, \quad r_1 = 2, \quad r_2 = -1.$$

Do the addition on the right-hand side above:

$$\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1(x+1) + a_2(x-2)}{(x-2)(x+1)}.$$

The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \displaystyle\int \frac{(x-1)}{(x^2 - x - 2)}\, dx.$

Solution: Recall: $\displaystyle\frac{(x-1)}{(x-2)(x+1)} = \frac{a_1(x+1) + a_2(x-2)}{(x-2)(x+1)}.$

The equation above implies:

$$x - 1 = a_1(x+1) + a_2(x-2)$$

To find a_1 evaluate the equation above at the root $r_1 = 2$,

$$1 = a_1(3) \quad \Rightarrow \quad a_1 = \frac{1}{3}.$$

To find a_2 evaluate the equation above at the root $r_2 = -1$,

$$-2 = a_2(-3) \quad \Rightarrow \quad a_2 = \frac{2}{3}.$$

We obtain $\displaystyle\frac{(x-1)}{(x-2)(x+1)} = \frac{1}{3}\,\frac{1}{(x-2)} + \frac{2}{3}\,\frac{1}{(x+1)}.$

The method of partial fractions (Non-repeated roots)

Example

Evaluate $I = \int \frac{(x-1)}{(x^2 - x - 2)}\, dx.$

Solution: Recall: $\frac{(x-1)}{(x-2)(x+1)} = \frac{1}{3}\frac{1}{(x-2)} + \frac{2}{3}\frac{1}{(x+1)}.$

The integral is now simple to evaluate,

$$I = \int \frac{(x-1)}{(x^2 - x - 2)}\, dx = \int \frac{1}{3}\frac{1}{(x-2)}\, dx + \int \frac{2}{3}\frac{1}{(x+1)}\, dx$$

We conclude that

$$I = \frac{1}{3}\ln|x-2| + \frac{2}{3}\ln|x+1| + c.$$

$\triangleleft$

The method of partial fractions (Non-repeated roots)

Theorem (Non-repeated roots - Heaviside cover method)

The rational function $\frac{p_k(x)}{(x-r_1)\cdots(x-r_n)}$, *with* $n > k$ *and all roots* $r_1, \cdots, r_n$ *different, can be written as*

$$\frac{p_k(x)}{(x-r_1)\cdots(x-r_n)} = \frac{a_1}{(x-r_1)} + \cdots + \frac{a_n}{(x-r_n)},$$

where the constants $a_1, \cdots, a_n$ *are given by*

$$a_1 = \frac{p_k(r_1)}{\prod_{j\neq 1}(r_1 - r_j)}, \quad \cdots \quad a_n = \frac{p_k(r_n)}{\prod_{j\neq n}(r_n - r_j)}.$$

Proof: $p_k(x) = a_1\left[\prod_{j\neq 1}(x - r_j)\right] + \cdots + a_n\left[\prod_{j\neq n}(x - r_j)\right].$ $\square$

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  - The general case.

The method of partial fractions (Repeated roots)

Example

Evaluate $I = \displaystyle\int \frac{(2x-1)}{(x^2-6x+9)}\,dx.$

Solution: First, find the zeros of the denominator,

$$x^2 - 6x + 9 = 0 \quad \Rightarrow \quad x_{\pm} = \frac{1}{2}\left[6 \pm \sqrt{36-36}\right] \quad \Rightarrow \quad x_{\pm} = 3.$$

Partial fraction problem: Find constants a_1 and a_2 such that

$$\frac{(2x-1)}{(x-3)^2} = \frac{a_1}{(x-3)} + \frac{a_2}{(x-3)^2}.$$

Do the addition on the right-hand side above:

$$\frac{(2x-1)}{(x-3)^2} = \frac{a_1(x-3)+a_2}{(x-3)^2}.$$

The method of partial fractions (Repeated roots)

Example

Evaluate $I = \int \frac{(2x-1)}{(x^2-6x+9)}\,dx.$

Solution: Recall: $\frac{(2x-1)}{(x-3)^2} = \frac{a_1(x-3)+a_2}{(x-3)^2}$. Then,

$$2x - 1 = a_1(x-3) + a_2.$$

To compute a_2 evaluate the expression above at $r = 3$,

$$5 = a_2.$$

To compute a_1 derivate the expression above, then evaluate at $r = 3$, (the evaluation at $r = 3$ is not needed in this case),

$$2 = a_1.$$

We conclude: $\frac{(2x-1)}{(x-3)^2} = \frac{2}{(x-3)} + \frac{5}{(x-3)^2}.$

The method of partial fractions (Repeated roots)

Example

Evaluate $I = \int \frac{(2x-1)}{(x^2-6x+9)}\,dx.$

Solution: Recall: $\frac{(2x-1)}{(x-3)^2} = \frac{2}{(x-3)} + \frac{5}{(x-3)^2}.$

The integral is now simple to evaluate,

$$I = \int \frac{(2x-1)}{(x^2-6x+9)}\,dx = \int \frac{2}{(x-3)}\,dx + \int \frac{5}{(x-3)^2}\,dx$$

We conclude that

$$I = 2\ln|x-3| - \frac{5}{(x-3)} + c. \qquad \triangleleft$$

The method of partial fractions (Repeated roots)

Theorem (Repeated roots)

The rational function $\dfrac{p_k(x)}{(x-r)^n}$*, with* $n > k$*, can be written as*

$$\frac{p_k(x)}{(x-r)^n} = \frac{a_1}{(x-r)} + \cdots + \frac{a_n}{(x-r)^n},$$

where a_i*, for* $i = 1, \cdots, n$*, is given by* $a_i = \dfrac{p_k^{(n-i)}(r)}{(n-i)!}$,

Proof: Taking common denominator on the right-hand side above,

$$p_k(x) = a_1\,(x-r)^{(n-1)} + a_2(x-r)^{(n-2)} + \cdots a_{(n-1)}(x-r) + a_n,$$

$$a_n = p_k(r),\; a_{(n-1)} = p'(r),\; \cdots\; a_2 = \frac{p^{(n-2)}(r)}{(n-2)!},\; a_1 = \frac{p^{(n-1)}(r)}{(n-1)!}.$$

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  - **The case** $\dfrac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}$, $b^2 - 4c < 0$ **(Complex roots).**
  - The general case.

The method of partial fractions (Complex roots)

Example

Evaluate $I = \int \frac{(x+1)^2}{(x^2+1)^2}\,dx.$

Solution: Find constants a_1, b_1 and a_2, b_2 such that

$$\frac{(x+1)^2}{(x^2+1)^2} = \frac{(a_1x+b_1)}{(x^2+1)} + \frac{(a_2x+b_2)}{(x^2+1)^2}.$$

$$\frac{(x+1)^2}{(x^2+1)^2} = \frac{(a_1x+b_1)(x^2+1)+(a_2x+b_2)}{(x^2+1)^2},$$

$$(x+1)^2 = (a_1x+b_1)(x^2+1)+(a_2x+b_2).$$

$$x^2+2x+1 = a_1x^3 + a_1x + b_1x^2 + b_1 + a_2x + b_2.$$

$$x^2+2x+1 = a_1x^3 + b_1x^2 + (a_1+a_2)x + (b_1+b_2).$$

The method of partial fractions (Complex roots)

Example

Evaluate $I = \int \frac{(x+1)^2}{(x^2+1)^2}\,dx.$

Solution: Recall:

$$x^2+2x+1 = a_1x^3 + b_1x^2 + (a_1+a_2)x + (b_1+b_2).$$

We conclude: $a_1 = 0$, $b_1 = 1$, $a_2 = 2$, and $b_2 = 0$. Hence,

$$I = \int \frac{(a_1x+b_1)}{(x^2+1)}\,dx + \int \frac{(a_2x+b_2)}{(x^2+1)^2}\,dx.$$

$$I = \int \frac{dx}{x^2+1} + \int \frac{2x\,dx}{(x^2+1)^2}.$$

We conclude that $I = \arctan(x) - \frac{1}{(x^2+1)} + c.$ $\triangleleft$

The method of partial fractions (Complex roots)

Theorem (Repeated roots)

The rational function $\frac{p_{(2n-1)}(x)}{(x+bx+c)^n}$, *with* $b^2-4c<0$, *can be written as*

$$\frac{p_{(2n-1)}(x)}{(x^2+bx+c)^n} = \frac{a_1x+b_1}{(x^2+bx+c)} + \cdots + \frac{a_nx+b_n}{(x^2+bx+c)^n}$$

for appropriate constants a_i, b_1 *for* $i=1,\cdots,n$.

Idea of the Proof:

Taking common denominator on the right-hand side above,

$$p_{(2n-1)}(x) = (a_1x+b_1)(x^2+bx+c)^{(n-1)} + \cdots + (a_nx+b_n).$$

Expanding the equation above one can find a system of equations for the coefficients.

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- The method of partial fractions.
  - The case $\frac{p_1(x)}{(x-r_1)(x-r_2)}$ $r_1 \neq r_2$ (Non-repeated roots).
  - The case $\frac{p_{(n-1)}(x)}{(x-r_1)^n}$. (Repeated roots).
  - The case $\frac{p_{(2n-1)}(x)}{(x^2+bx+c)^n}$, $b^2-4c<0$ (Complex roots).
  - **The general case.**

The method of partial fractions (General case)

Remarks:

- Consider a general rational function $\dfrac{r_k(x)}{q_n(x)}$, with $k < n$.
- Express the denominator, q, as a product of factors $(x - r_i)^{m_i}$ and $(x^2 + b_i x + c_i)^{\ell_i}$, with r_i roots of q_n, and $b_i^2 - 4c_i < 0$.
- The partial fraction decomposition for $\dfrac{r_k}{q_n}$ is the addition of the partial fraction decomposition for each factor in q.

The method of partial fractions (General case)

Example

Evaluate $I = \displaystyle\int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x}\,dx$.

Solution: The partial fraction decomposition is:

$$\frac{6x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x} = \frac{(ax+b)}{(x^2+1)} + \frac{c}{(x-2)} + \frac{d}{x}$$

$$6x^3 - 8x^2 + 5x - 6 = (ax+b)(x-2)x + c(x^2+1)x + d(x^2+1)(x-2)$$

$$= ax^3 - 2ax^2 + bx^2 - 2bx + cx^3 + cx + dx^3 - 2dx^2 + dx - 2d$$

$$= (a + c + d)x^3 + (-2a + b - 2d)x^2 + (-2b + c + d)x - 2d$$

$$a + c + d = 6, \quad -2a + b - 2d = -8, \quad 5 = -2b + c + d \quad d = 3.$$

The method of partial fractions (General case)

Example

Evaluate $I = \int \frac{9x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x}\,dx.$

Solution: Recall:

$$a + c + d = 6, \quad -2a + b - 2d = -8, \quad 5 = -2b + c + d \quad d = 3.$$

$$a + c = 3, \quad 2a - b = 2, \quad -2b + c = 2.$$

$$c = 3 - a \;\Rightarrow\; -2b + 3 - a = 2 \;\Rightarrow\; a = 1 - 2b \;\Rightarrow\; 2 - 4b - b = 2.$$

Hence $b = 0$, and then $a = 1$ and $c = 2$. We conclude,

$$I = \int \frac{6x^3 - 8x^2 + 5x - 6}{(x^2+1)(x-2)x}\,dx = \int \Big[\frac{x}{(x^2+1)} + \frac{2}{(x-2)} + \frac{3}{x}\Big]\,dx$$

$$I = \frac{1}{2}\ln(x^2+1) + 2\ln|x-2| + 3\ln|x| + c. \qquad \triangleleft$$