Review for Exam 2.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to homeworks.
- Exam covers: 7.4, 7.6, 7.7, 8-IT, 8.1, 8.2.
  - Solving differential equations (7.4).
  - Inverse trigonometric functions (7.6).
  - Hyperbolic functions (7.7).
  - Integration techniques (8-IT).
  - Integration by parts (8.1).
  - Trigonometric integrals (8.2).
- Section not covered:
  - Trigonometric substitutions (8.3).

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  - **Solving differential equations (7.4).**
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Solving differential equations (7.4)

Remark: Typical problems in this section:

(1) Find the function y solution of $y' = \dfrac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

(2) The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below $1/8$ the intensity at the surface?

Solving differential equations (7.4)

Example

Find the function y solution of $y' = \dfrac{\sin(x)}{4y}$ and $y(0) = -\sqrt{2}$.

Solution:

$$4y\,y' = \sin(x) \quad \Rightarrow \quad \int 4y(x)\,y'(x)\,dx = \int \sin(x)\,dx.$$

The substitution $u = y(x)$, with $du = y'(x)\,dx$, implies

$$4\int u\,du = \int \sin(x)\,dx \;\Rightarrow\; 2u^2 = -\cos(x) + c,$$

Therefore, $y^2(x) = (-\cos(x) + c)/2$. The condition $y(0) < 0$, implies $y(x) = -\sqrt{c - \cos(x)}/\sqrt{2}$. Furthermore,

$$-\sqrt{2} = -\frac{\sqrt{c-1}}{\sqrt{2}} \;\Rightarrow\; 2 = \sqrt{c-1} \;\Rightarrow\; c = 5.$$

We conclude that $y = -\sqrt{5 - \cos(x)}/\sqrt{2}$. $\triangleleft$

Solving differential equations (7.4)

Example

The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below 1/8 the intensity at the surface?

Solution: Integrate the differential equation,

$$\int \frac{L'(x)}{L(x)}\, dx = -k \int dx, \quad u = L(x), \quad du = L'(x)\, dx$$

$$\int \frac{du}{u} = -k \int dx \;\Rightarrow\; \ln(u) = -kx + c \;\Rightarrow\; L(x) = e^{-kx+c}.$$

Since $L(x) = e^{-kx} e^{c}$, calling $L_0 = e^{c}$, we get the solution

$$L(x) = L_0\, e^{-kx}.$$

Solving differential equations (7.4)

Example

The intensity $L(x)$ of light x feet beneath the surface of the ocean satisfies the equation $L' = -kL$, for some $k > 0$. If diving at 15 ft cuts the light intensity in half, how deep the light intensity falls below 1/8 the intensity at the surface?

Solution: Recall: $L(x) = L_0\, e^{-kx}$. Now the first condition implies

$$\frac{L_0}{2} = L(15) = L_0\, e^{-15k} \;\Rightarrow\; e^{-15k} = \frac{1}{2} \;\Rightarrow\; -15k = -\ln(2)$$

so we conclude that $k = \ln(2)/15$. The second condition implies

$$\frac{L_0}{8} = L_0\, e^{-kx_1} \;\Rightarrow\; e^{-kx_1} = \frac{1}{8} \;\Rightarrow\; -kx_1 = -\ln(8).$$

Using the value $k = \ln(2)/15$, we get

$$x_1 = \ln(8)\, \frac{15}{\ln(2)} \quad\Rightarrow\quad x_1 = 3(15) \quad\Rightarrow\quad x_1 = 45. \qquad \triangleleft$$

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Inverse trigonometric functions (7.6)

Notation: In the literature is common the notation $\sin^{-1} = \arcsin$, and similar for the rest of the trigonometric functions.

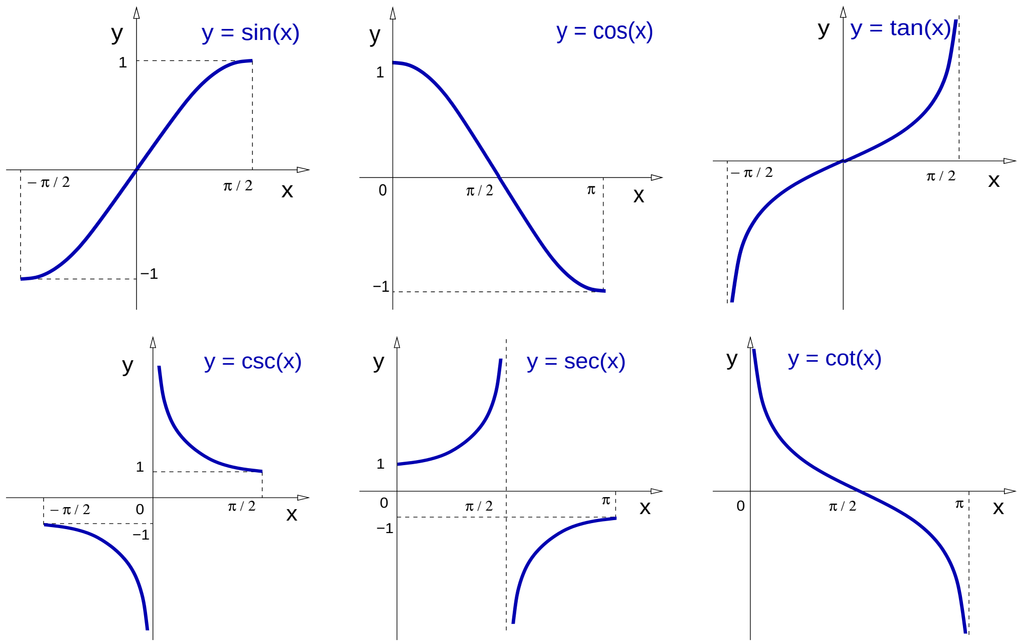
Do not confuse $\dfrac{1}{\sin(x)} \neq \sin^{-1}(x) = \arcsin(x)$.

Remark: sin, cos have simple values at particular angles.

θ	$\sin(\theta)$	$\cos(\theta)$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

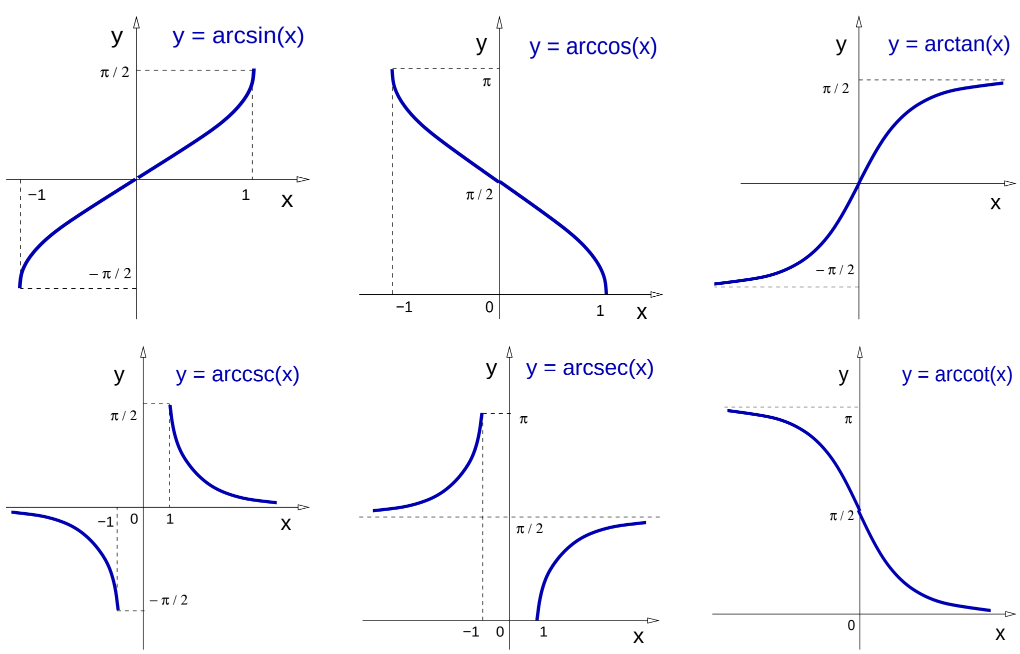
Inverse trigonometric functions (7.6)

Remark: On certain domains the trigonometric functions are invertible.



Inverse trigonometric functions (7.6)

Remark: The graph of the inverse function is a reflection of the original function graph about the $y = x$ axis.



Inverse trigonometric functions (7.6)

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$
$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \operatorname{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$
$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| \geqslant 1.$$

Recall $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}$, $\tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

$$\tan'(y) = 1 + \tan^2(y), \;\; y = \arctan(x), \;\; \Rightarrow \;\; \arctan'(x) = \frac{1}{1+x^2}.$$

Inverse trigonometric functions (7.6)

Remark: Typical problems in this section:

(1) Sketch the graphs of

$$y(x) = \sec(x), \qquad z(x) = \operatorname{arcsec}(x).$$

State the respective domains and ranges.

(2) Evaluate $\cos\big(\arcsin(1/\sqrt(2))\big)$.

(3) Evaluate $\sec\big(\arctan(-2/3)\big)$.

(4) Find y' for $y(x) = \arctan(3x^2)$.

(5) Find $I = \displaystyle\int \frac{dx}{\sqrt{2-x^2}}$.

Inverse trigonometric functions (7.6)

Example

Evaluate $\sec\big(\arctan(-2/3)\big)$.

Solution: We only need the relation between sec and tan,

$$\sec^2(\theta) = \tan^2(\theta) + 1.$$

Then holds $\sec(\theta) = \pm\sqrt{\tan^2(\theta) + 1}$. We need to find the correct sign: $\theta = \arctan(-2/3) \in (-\pi/2, 0)$. Since $\sec(\theta) = 1/\cos(\theta)$, we conclude that $\sec(\theta) > 0$. Hence

$$\sec(\arctan\Big(-\frac{2}{3}\Big)) = \sqrt{\tan^2(\arctan\Big(-\frac{2}{3}\Big)) + 1} = \sqrt{\frac{4}{9} + 1} = \sqrt{\frac{13}{9}}.$$

We conclude that $\sec(\arctan(-2/3)) = \sqrt{13}/3$. $\lhd$

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Hyperbolic functions (7.7)

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Hyperbolic functions (7.7)

Remark: Typical problems in this section:

(1) Prove the identities: $\cosh^2(x) - \sinh^2(x) = 1$, and

$$\cosh(2x) = \cosh^2(x) + \sinh^2(x), \quad \sinh(2x) = 2\sinh(x)\cosh(x)),$$

$$\cosh^2(x) = \frac{1}{2}\big(1 + \cosh(2x)\big), \qquad \sinh^2(x) = \frac{1}{2}\big(-1 + \cosh(2x)\big).$$

(2) Know the derivatives and integrals of hyperbolic functions.

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Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) $\displaystyle\int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

(2) $\displaystyle\int_1^8 \frac{dx}{x^2-2x+50}$.

(3) $\displaystyle\int x^3 \ln(x)\,dx$.

(4) $\displaystyle\int x^2 e^{2x}\,dx$.

(5) $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$.

(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}, \ |x|<5$.

(7) $\displaystyle\int \cot^3(x)\,dx$.

(8) $\displaystyle\int \sin^4(x)\,dx$.

(9) $\displaystyle\int x^3\cos(x)\,dx$.

(10) $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos(2x)}\,dx$.

(11) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)}\,dx$.

(12) $\displaystyle\int \frac{2^{\ln(x)}}{x}\,dx$.

Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(1) $\displaystyle\int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$. Split the integral and do two substitutions.

(2) $\displaystyle\int_1^8 \frac{dx}{x^2-2x+50}$. Complete the square and recall the arctan′.

(3) $\displaystyle\int x^3 \ln(x)\,dx$. Three integrations by parts.

(4) $\displaystyle\int x^2 e^{2x}\,dx$. Two integrations by parts.

(5) $\displaystyle\int \frac{dx}{\sqrt{8x-x^2}}$. Complete the square and recall arcsin′.

(6) $\displaystyle\int \frac{dx}{\sqrt{25-x^2}}$, $|x| < 5$. Substitution and recall arcsin′.

Sections 8-IT, 8.1, 8.2

Remark: Evaluate the following integrals:

(7) $\displaystyle\int \cot^3(x)\,dx$. Write using sin, cos and substitution.

(8) $\displaystyle\int \sin^4(x)\,dx$. Double angle formula, twice.

(9) $\displaystyle\int x^3 \cos(x)\,dx$. Integrations by parts, three times.

(10) $\displaystyle\int_{-\pi/2}^{\pi/2} \sqrt{1-\cos(2x)}\,dx$. Double angle formula, cancel $\sqrt{\ }$.

(11) $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2(x)}{\tan(x)}\,dx$. Write using sin and cos, and substitution.

(12) $\displaystyle\int \frac{2^{\ln(x)}}{x}\,dx$. Substitution.

Example

Evaluate $I = \int \frac{(1+x)\,dx}{\sqrt{1-2x^2}}$.

Solution: Split the integral: $I = \int \frac{dx}{\sqrt{1-2x^2}} + \int \frac{x\,dx}{\sqrt{1-2x^2}}$.

For the first integral substitute $y = \sqrt{2}\,x$, then $dy = \sqrt{2}\,dx$.

$$I_1 = \int \frac{dx}{\sqrt{1-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{dy}{\sqrt{1-y^2}} = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}\,x) + c.$$

For the second integral substitute $u = 1 - 2x^2$, then $du = -4x\,dx$.

$$I_2 = -\frac{1}{4} \int \frac{du}{\sqrt{u}} = -\frac{1}{4}(2\sqrt{u}) + c = -\frac{1}{2}\sqrt{1-2x^2} + c.$$

We conclude: $I = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}\,x) - \frac{1}{2}\sqrt{1-2x^2} + c.$ $\triangleleft$

Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int \frac{dx}{\sqrt{8x - x^2}}$.

Solution: Complete the square and recall $\arcsin'$.

$$I = \int \frac{dx}{\sqrt{-x^2 + 2(4x)}} = \int \frac{dx}{\sqrt{-x^2 + 2(4x) - 4^2 + 4^2}},$$

$$I = \int \frac{dx}{\sqrt{4^2 - (x^2 - 2(4x) + 4^2)}} = \int \frac{dx}{\sqrt{4^2 - (x-4)^2}}$$

$$I = \frac{1}{4} \int \frac{dx}{\sqrt{1 - [(x-4)/4]^2}}.$$

Substitute $u = (x-4)/4$, then $du = dx/4$.

$$I = I \int \frac{du}{\sqrt{1-u^2}} = \arcsin(u) + c \;\Rightarrow\; I = \arcsin\Big(\frac{(x-4)}{4}\Big) + c.$$

Sections 8-IT, 8.1, 8.2

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos(2x)}\, dx.$

Solution: Double angle formula, cancel $\sqrt{\ }$.

Recall: $\sin^2(\theta) = [1 - \cos(2\theta)]/2$. Hence,

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{2\sin^2(x)}\, dx = \sqrt{2} \int_{-\pi/2}^{\pi/2} |\sin(x)|\, dx.$$

Since $\sin(x) < 0$ for $x \in (-\pi/2, 0)$,

$$I = -\sqrt{2} \int_{-\pi/2}^{0} \sin(x)\, dx + \sqrt{2} \int_{0}^{\pi/2} \sin(x)\, dx.$$

$$I = \sqrt{2}\cos(x)\Big|_{-\pi/2}^{0} - \sqrt{2}\cos(x)\Big|_{0}^{\pi/2} = \sqrt{2}(1-0) - \sqrt{2}(0-1) = 2\sqrt{2}.$$