Trigonometric substitutions (Sect. 8.3)

- Substitutions to cancel the square root
- Integrals involving $\sqrt{a^2 - x^2}$: Use $x = a\sin(\theta)$.
- Integrals involving $\sqrt{a^2 + x^2}$: Use $x = a\tan(\theta)$.
- Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.

Substitutions to cancel the square root

Remark: Integrals involving $\sqrt{a^2 - x^2}$ can be found with the substitution $x = a\sin(\theta)$. Indeed,

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2(\theta)} = |a|\sqrt{1 - \sin^2(\theta)} = |a|\,|\cos(\theta)|.$$

We conclude that $\sqrt{a^2 - x^2} = |a|\,|\cos(\theta)|$.

Notice: The substitution $x = a\cos(\theta)$ also works.

Remark: We have used Pythagoras Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

To compute dx we will need the following derivatives:

$$\sin'(\theta) = \cos(\theta), \qquad \cos'(\theta) = -\sin(\theta).$$

Substitutions to cancel the square root

Recall:

$$\sec^2(\theta) = \tan^2(\theta) + 1 = \tan'(\theta), \quad \sec'(\theta) = \sec(\theta)\,\tan(\theta).$$

Remark: Integrals involving $\sqrt{a^2 + x^2}$ can be found with the substitution $x = a\tan(\theta)$. Indeed,

$$\sqrt{a^2 + x^2} = \sqrt{a^2 + a^2\tan^2(\theta)} = |a|\sqrt{1 + \tan^2(\theta)} = |a|\,|\sec(\theta)|.$$

Hence, $\sqrt{a^2 + x^2} = |a|\,|\sec(\theta)|$, and $dx = a\,\sec^2(\theta)\,d\theta$.

Remark: Integrals involving $\sqrt{x^2 - a^2}$ can be found with the substitution $x = a\sec(\theta)$. Indeed,

$$\sqrt{x^2 - a^2} = \sqrt{a^2\sec^2(\theta) - a^2} = |a|\sqrt{\sec^2(\theta) - 1} = |a|\,|\tan(\theta)|.$$

Hence, $\sqrt{x^2 - a^2} = |a|\,|\tan(\theta)|$, and $dx = a\,\sec(\theta)\,\tan(\theta)\,d\theta$.

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Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-5}^{5} \sqrt{25 - x^2}\, dx$.

Solution: Recall Pythagoras Theorem:

$$\sin^2(\theta) + \cos^2(\theta) = 1 \quad \Rightarrow \quad |\cos(\theta)| = \sqrt{1 - \sin^2(\theta)}.$$

Substitute: $x = 5\sin(\theta)$, then $dx = 5\cos(\theta)\, d\theta$. Hence

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - 25\sin^2(\theta)}\, \big(5\,\cos(\theta)\, d\theta\big),$$

$$I = 5 \int_{-\pi/2}^{\pi/2} \sqrt{25\big[1 - \sin^2(\theta)\big]}\, \cos(\theta)\, d\theta,$$

$$I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta.$$

Integrals involving $\sqrt{a^2 - x^2}$

Example

Evaluate $I = \int_{-\pi/2}^{\pi/2} \sqrt{25 - x^2}\, dx$.

Solution: Recall: $I = 5^2 \int_{-\pi/2}^{\pi/2} \sqrt{1 - \sin^2(\theta)}\, \cos(\theta)\, d\theta$.

$$I = 5^2 \int_{-\pi/2}^{\pi/2} |\cos(\theta)|\, \cos(\theta)\, dx = 5^2 \int_{-\pi/2}^{\pi/2} \cos^2(\theta)\, d\theta.$$

$$I = \frac{25}{2} \int_{-\pi/2}^{\pi/2} \big[1 + \cos(2\theta)\big]\, d\theta = \frac{25}{2}\Big(\theta + \frac{\sin(2\theta)}{2}\Big)\Big|_{-\pi/2}^{\pi/2}.$$

We conclude that $I = \dfrac{25\pi}{2}$. ◁

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- Integrals involving $\sqrt{x^2 - a^2}$: Use $x = a\sec(\theta)$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \int \frac{x^3\,dx}{\sqrt{x^2+4}}$ for $x \in \mathbb{R}$.

Solution: The only thing that matters to choose the substitution is the argument of the square root.

In this case we need to recall $1 + \tan^2(\theta) = \sec^2(\theta)$.

Hence the substitution $x = 2\tan(\theta)$, for $\theta \in (-\pi/2, \pi/2)$.

Then, $dx = 2\tan'(\theta)\,d\theta$, that is $dx = 2\sec^2(\theta)\,d\theta$.

$$I = \int \frac{2^3\tan^3(\theta)}{\sqrt{4\tan^2(\theta)+4}}\,2\sec^2(\theta)\,d\theta = 16\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sqrt{4\left[\tan^2(\theta)+1\right]}}\,d\theta.$$

$$I = \frac{16}{2}\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{|\sec(\theta)|}\,d\theta = 8\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta.$$

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$, for $x \in \mathbb{R}$.

Solution: Recall: $I = 8 \displaystyle\int \frac{\tan^3(\theta)\,\sec^2(\theta)}{\sec(\theta)}\,d\theta$.

$$I = 8 \int \tan^3(\theta)\,\sec(\theta)\,d\theta = 8 \int \tan^2(\theta)\,\tan(\theta)\sec(\theta)\,d\theta.$$

Now recall $\tan(\theta)\sec(\theta) = \sec'(\theta)$, and that $\tan^2(\theta) = \sec^2(\theta) - 1$.

$$I = 8 \int \big[\sec^2(\theta) - 1\big]\,\sec'(\theta)\,d\theta.$$

Substitute: $u = \sec(\theta)$, $u \in [1, \infty)$, hence $du = \sec'(\theta)\,d\theta$.

Integrals involving $\sqrt{a^2 + x^2}$

Example

Evaluate $I = \displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$, for $x \in \mathbb{R}$.

Solution: Recall: $I = 8 \displaystyle\int \big[\sec^2(\theta) - 1\big]\,\sec'(\theta)\,d\theta$ together with the substitution: $u = \sec(\theta)$, so $du = \sec'(\theta)\,d\theta$.

$$I = 8 \int (u^2 - 1)\,du = 8\Big(\frac{u^3}{3} - u + c\Big).$$

Substitute back $u = \sec(\theta)$, that is,

$$I = \frac{8}{3}\,\sec^3(\theta) - 8\sec(\theta) + c.$$

Still substitute back $x = 2\tan(\theta)$, that is $\theta = \arctan(x/2)$, hence

$$I = \frac{8}{3}\,\sec^3\big(\arctan(x/2)\big) - 8\sec(\arctan(x/2) + c. \qquad \triangleleft$$

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Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 3$.

Solution: Once again, it only matters what is in the square root.

We need to recall $\sec^2(\theta) = \tan^2(\theta) + 1$, so $\sec^2(\theta) - 1 = \tan^2(\theta)$.

Substitute $x = 3\sec(\theta)$, $\theta \in (0, \pi/2)$, hence $dx = 3\sec'(\theta)\,d\theta$.

$$I = \int \frac{2}{9\sec^2(\theta)\,\sqrt{9\sec^2(\theta) - 9}}\,3\sec'(\theta)\,d\theta.$$

$$I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\,\sqrt{\sec^2(\theta) - 1}} = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)|\tan(\theta)|}.$$

$\sec'(\theta) = \sec(\theta)\,\tan(\theta)$, and $\theta \in (0, \pi/2)$ implies $\tan(\theta) > 0$.

Integrals involving $\sqrt{x^2 - a^2}$

Example

Evaluate $I = \int \frac{2\,dx}{x^2\sqrt{x^2-9}}$, for $x > 3$.

Solution: So, $I = \frac{2}{9}\int \frac{\sec'(\theta)\,d\theta}{\sec^2(\theta)\tan(\theta)}$, and $\sec'(\theta) = \sec(\theta)\,\tan(\theta)$.

$$I = \frac{2}{9}\int \frac{\sec(\theta)\tan(\theta)\,d\theta}{\sec^2(\theta)\,\tan(\theta)} = \frac{2}{9}\int \frac{d\theta}{\sec(\theta)} = \frac{2}{9}\int \cos(\theta)\,d\theta.$$

$$I = \frac{2}{9}\sin(\theta) + c.$$

Substitute back $x = 3\sec(\theta)$, that is, $\theta = \operatorname{arcsec}(x/3)$.

$$I = \frac{2}{9}\sin\big(\operatorname{arcsec}(x/3)\big) + c.$$

$\triangleleft$