Integration by parts (Sect. 8.1)

- Integral form of the product rule.
- Exponential and logarithms.
- Trigonometric functions.
- Definite integrals.
- Substitution and integration by parts.

Integral form of the product rule

Remark: The integration by parts formula is an integral form of the product rule for derivatives: $(fg)' = f'\,g + f\,g'$.

Theorem

For all differentiable functions $g, f : \mathbb{R} \to \mathbb{R}$ holds

$$\int f(x)\,g'(x)\,dx = f(x)\,g(x) - \int f'(x)\,g(x)\,dx.$$

Proof: Integrate the product rule $f\,g' = (fg)' - f'\,g$, and use the Fundamental Theorem of Calculus in $\int (fg)'\,dx = fg$. □

Notation: It is common to write $\int u\,dv = uv - \int v\,du$, where

$$u = f(x), \quad dv = g'(x)\,dx, \quad \text{and} \quad v = g(x), \quad du = f'(x)\,dx.$$

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Exponentials and logarithms

Example

Evaluate $I = \int x\, e^{2x}\, dx$.

Solution: Recall: $\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx.$

We need to choose the functions f and g.

They should be chosen in a way that the right-hand side above is simpler to integrate than the original left-hand side.

$$f(x) = x, \quad g'(x) = e^{2x} \quad \Rightarrow \quad f'(x) = 1, \quad g(x) = \frac{e^{2x}}{2}.$$

$$\int x\, e^{2x}\, dx = \frac{x\, e^{2x}}{2} - \int \frac{e^{2x}}{2}\, dx = \frac{x\, e^{2x}}{2} - \frac{e^{2x}}{4} + c.$$

We conclude $I = \frac{e^{2x}}{4}(2x - 1) + c.$ $\triangleleft$

Exponentials and logarithms

Example

Evaluate $I = \int x\, e^{2x}\, dx$.

Solution: Recall: $\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$.

We need to choose the functions f and g.

They should be chosen in a way that the right-hand side above is simpler to integrate than the original left-hand side. Wrong choice:

$$f(x) = e^{2x}, \quad g'(x) = x \quad \Rightarrow \quad f'(x) = 2\, e^{2x}, \quad g(x) = \frac{x^2}{2}.$$

$$\int x\, e^{2x}\, dx = \frac{x^2\, e^{2x}}{2} - \int x^2 e^{2x}\, dx.$$

This is the wrong choice. ◁

Exponentials and logarithms

Remark: We use now the $\int u\, dv = uv - \int v\, du$ notation.

Example

Evaluate $I = \int \ln(x)\, dx$.

Solution: We need to choose u and v:

$$u = \ln(x), \quad dv = dx \quad \Rightarrow \quad du = \frac{dx}{x}, \quad v = x.$$

$$\int \ln(x)\, dx = x\, \ln(x) - \int x\, \frac{dx}{x} = x\, \ln(x) - \int dx.$$

We conclude that $I = x\, \ln(x) - x + c$. ◁

Exponentials and logarithms

Remark: Integration by parts can be used more than once.

Example

Evaluate $I = \int x^2 \, e^x \, dx$.

Solution: $u = x^2, \quad dv = e^x \, dx \quad \Rightarrow \quad du = 2x \, dx, \quad v = e^x.$

$$\int x^2 \, e^x \, dx = x^2 \, e^x - 2 \int x \, e^x \, dx$$

We integrate by parts one more time.

$$u = x, \quad dv = e^x \, dx \quad \Rightarrow \quad du = dx, \quad v = e^x.$$

$$\int x^2 \, e^x \, dx = x^2 \, e^x - 2 \Big[x \, e^x - \int e^x \, dx \Big]$$

We conclude that $I = x^2 \, e^x - 2x \, e^x + 2e^x + c$. $\triangleleft$

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Trigonometric functions

Example

Evaluate $I = \int x \sin(x)\, dx$.

Solution: We choose u, v so that we simplify the integral:

$$u = x, \quad dv = \sin(x)\, dx, \quad \Rightarrow \quad du = dx, \quad v = -\cos(x).$$

$$I = -x \cos(x) - \int [-\cos(x)]\, dx.$$

$$I = -x \cos(x) + \int \cos(x)\, dx.$$

We conclude that $I = -x \cos(x) + \sin(x) + c$. $\triangleleft$

Trigonometric functions

Example

Evaluate $I = \int e^{ax} \sin(x)\, dx$.

Solution: In this case we need to integrate by parts twice.

$$u = e^{ax}, \quad dv = \sin(x)\, dx \quad \Rightarrow \quad du = a\, e^{ax}\, dx, \quad v = -\cos(x).$$

$$I = -e^{ax} \cos(x) + a \int e^{ax} \cos(x)\, dx.$$

$$u = e^{ax}, \quad dv = \cos(x)\, dx \quad \Rightarrow \quad du = a\, e^{ax}\, dx, \quad v = \sin(x).$$

$$I = -e^{ax} \cos(x) + a \Big[e^{ax} \sin(x) - a \int e^{ax} \sin(x)\, dx \Big].$$

$$\int e^{ax} \sin(x)\, dx = -e^{ax} \cos(x) + a\, e^{ax} \sin(x) - a^2 \int e^{ax} \sin(x)\, dx.$$

Trigonometric functions

Example

Evaluate $I = \int e^{ax} \sin(x)\,dx$.

Solution: Recall:

$$\int e^{ax} \sin(x)\,dx = -e^{ax}\cos(x) + a\,e^{ax}\sin(x) - a^2 \int e^{ax}\sin(x)\,dx.$$

Remark: The last term on the right-hand side is proportional to the negative of the left-hand side. So, for all $a \neq 0$ holds

$$(1 + a^2) \int e^{ax} \sin(x)\,dx = -e^{ax}\cos(x) + a\,e^{ax}\sin(x).$$

We then conclude that

$$\int e^{ax} \sin(x)\,dx = \frac{e^{ax}}{(1+a^2)}\big[-\cos(x) + a\sin(x)\big]. \qquad \lhd$$

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Definite integrals

Remark: Integration by parts can be used with definite integrals.

Theorem

For all differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$ holds

$$\int_a^b f(x)\, g'(x)\, dx = \big[f(x)\, g(x)\big]\Big|_a^b - \int_a^b f'(x)\, g(x)\, dx.$$

Example

Evaluate $I = \displaystyle\int_0^\pi e^{ax} \sin(x)\, dx$.

Solution: Use integrations by parts and evaluate the result:

$$\int_0^\pi e^{ax} \sin(x)\, dx = \Big[\frac{e^{ax}}{(1+a^2)}\big[-\cos(x) + a \sin(x)\big]\Big]\Big|_0^\pi$$

$$\int_0^\pi e^{ax} \sin(x)\, dx = \frac{(e^{a\pi}+1)}{(1+a^2)}. \qquad \triangleleft$$

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- **Substitution and integration by parts.**

Substitution and integration by parts

Remark: Substitution and integration by parts can be used on the same integral.

Example

Evaluate $I = \int \cos(\ln(x))\,dx$.

Solution: We start with the substitution $y = \ln(x)$, and $dy = \dfrac{dx}{x}$.

$$dx = x\,dy = e^y\,dy.$$

The integral is $I = \int \cos(y)\,e^y\,dy$, and we integrate by parts.

If $u = e^y$, $dv = \cos(y)\,dy$, then $du = e^y\,dy$, $v = \sin(y)$,

$$\int e^y \cos(y)\,dy = e^y\,\sin(y) - \int e^y\,\sin(y)\,dy.$$

Substitution and integration by parts

Example

Evaluate $I = \int \cos(\ln(x))\,dx$.

Solution: Recall: $\int e^y \cos(y)\,dy = e^y\,\sin(y) - \int e^y\,\sin(y)\,dy$.

One more integration by parts,

$$u = e^y, \quad dv = \sin(y)\,dy, \quad \Rightarrow \quad du = e^y\,dy, \quad v = -\cos(y).$$

$$\int e^y \cos(y)\,dy = e^y\,\sin(y) - \Big[-e^y\,\cos(y) + \int e^y\,\cos(y)\,dy\Big].$$

$$2\int e^y \cos(y)\,dy = e^y\,\sin(y) + e^y\,\cos(y)$$

$$\int e^y \cos(y) = \frac{e^y}{2}\big[\sin(y) + \cos(y)\big].$$

We conclude: $\int \cos(\ln(x))\,dx = \dfrac{x}{2}\big[\sin(\ln(x))) + \cos(\ln(x))\big]$. $\triangleleft$