Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- Polynomial division.
- Multiplying by 1.

Substitution rule

Theorem

For every differentiable functions $f, u : \mathbb{R} \to \mathbb{R}$ holds,

$$\int f(u(x))\, u'(x)\, dx = \int f(y)\, dy.$$

Proof: This is the integral form of the chain rule for derivatives. Let F be a primitive of f, that is, $F' = f$. Then

$$\big(F(u)\big)' = F'(u)\, u' = f(u)\, u'.$$

Integrate the equation above,

$$\int f(u(x))\, u'(x)\, dx = \int \frac{d(F(u))}{dx}(x)\, dx = F(u(x)).$$

Denoting $y = u(x)$, we get

$$\int f(u(x))\, u'(x)\, dx = F(y) = \int F'(y)\, dy = \int f(y)\, dy. \qquad \square$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx.$

Solution: The argument in the tangent function is has an x^2, and in the integral appears the factor $x\, dx$. We try the substitution

$$u = \tan(2x^2), \qquad du = \frac{1}{\cos^2(2x^2)}\,(4x)\, dx.$$

This substitution will simplify the integration if $\cos^2(2x^2)$ can be related to $[1+\cos(4x^2)]$. And this is the case, since

$$\cos^2(\theta) = \frac{1}{2}\big(1+\cos(2\theta)\big), \qquad \theta = 2x^2,$$

$$I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx = \int \frac{3x\, e^{\tan(2x^2)}}{2\cos^2(2x^2)}\, dx = \frac{3}{2}\int e^u\, \frac{du}{4}.$$

Substitution rule

Example

Evaluate $I = \int \frac{3x\, e^{\tan(2x^2)}}{[1+\cos(4x^2)]}\, dx.$

Solution: Recall: $I = \frac{3}{2}\int e^u\, \frac{du}{4}$, with $u = \tan(2x^2)$.

$$I = \frac{3}{8}\int e^u\, du = \frac{3}{8}\, e^u + c.$$

We now substitute back with $u = \tan(2x^2)$,

$$I = \frac{3}{8}\, e^{\tan(2x^2)} + c.$$

◁

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- **Completing the square.**
- Trigonometric identities.
- Polynomial division.
- Multiplying by 1.

Completing the square

Example

Evaluate $I = \displaystyle\int \frac{dx}{(x+3)\sqrt{x^2+6x+4}}$.

Solution: The idea is to rewrite the function inside the square root:

$$x^2 + 6x + 4 = x^2 + 2\Big(\frac{6}{2}\Big)\, x + 4 = x^2 + 2(3x) + 4$$

$$x^2 + 6x + 4 = [x^2 + 2(3x) + 9] - 9 + 4 = (x+3)^2 - 5.$$

$$I = \int \frac{dx}{(x+3)\sqrt{(x+3)^2 - 5}} \qquad u = x + 3, \quad du = dx$$

$$I = \int \frac{du}{u\sqrt{u^2 - 5}} = \frac{1}{\sqrt{5}} \operatorname{arcsec}\Big(\frac{|u|}{\sqrt{5}}\Big) + c.$$

We obtain $I = \dfrac{1}{\sqrt{5}} \operatorname{arcsec}\Big(\dfrac{|x+3|}{\sqrt{5}}\Big) + c.$ $\triangleleft$

Completing the square

Remark: Sometimes completing the square is not needed.

Example

Evaluate $I = \int \frac{(x+3)\,dx}{\sqrt{x^2+6x+4}}$.

Solution: Since the factor $(x+3)$ is in the numerator, instead of the denominator, substitution will work:

$$u = x^2 + 6x + 4, \qquad du = (2x+6)\,dx = 2(x+3)\,dx.$$

$$I = \int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int u^{-1/2}\,du = \frac{1}{2}\big(2u^{1/2}\big) + c.$$

We conclude that $I = \sqrt{x^2+6x+4} + c$. $\triangleleft$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- **Trigonometric identities.**
- Polynomial division.
- Multiplying by 1.

Trigonometric identities

Example

Evaluate $I = \int \big[\sec(x) + \tan(x)\big]^2\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$f(x) = \big[\sec(x) + \tan(x)\big]^2 = \Big[\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\Big]^2 = \frac{(1+\sin(x))^2}{\cos^2(x)}.$$

$$f(x) = \frac{1 + 2\sin(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)}.$$

$$\frac{\sin^2(x)}{\cos^2(x)} = \frac{1-\cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

Trigonometric identities

Example

Evaluate $I = \int \big[\sec(x) + \tan(x)\big]^2\, dx$.

Solution: Recall: $f(x) = \big[\sec(x) + \tan(x)\big]^2$,

$$f(x) = \frac{1}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\cos^2(x)} \text{ and } \frac{\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} - 1.$$

$$f(x) = \frac{2}{\cos^2(x)} + \frac{2\sin(x)}{\cos^2(x)} - 1.$$

$$I = \int \frac{2\,dx}{\cos^2(x)} + \int \frac{2\sin(x)\,dx}{\cos^2(x)} - \int dx. \qquad u = \cos(x), \quad du = -\sin(x)\,dx.$$

$$I = 2\tan(x) - 2\int \frac{du}{u^2} - x + c \;\Rightarrow\; I = 2\tan(x) + \frac{2}{\cos(x)} - x + c.$$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- **Polynomial division.**
- Multiplying by 1.

Polynomial division

Example

Evaluate $I = \int \frac{4x^2 - 7}{2x + 3}\, dx$.

Solution: The degree of the polynomial in the numerator is greater or equal the degree of the polynomial in the denominator.
In this case it is convenient to do the division:

$$\begin{array}{r|l} & 2x - 3 \\ \hline 2x+3) & 4x^2 \qquad - 7 \\ & -4x^2 - 6x \\ \hline & -6x - 7 \\ & 6x + 9 \\ \hline & 2 \end{array} \quad \Rightarrow \quad \frac{4x^2 - 7}{2x + 3} = 2x - 3 + \frac{2}{2x + 3}.$$

$$I = \int (2x - 3)\, dx + \int \frac{2\, dx}{2x + 3} \quad \Rightarrow \quad I = x^2 - 3x + \ln(2x + 3) + c.$$

Integration techniques (Supp. Material 8-IT)

- Substitution rule.
- Completing the square.
- Trigonometric identities.
- Polynomial division.
- **Multiplying by 1.**

Multiplying by 1

Example

Evaluate $I = \int \frac{dx}{1 + \sin(x)}$.

Solution:

$$I = \int \frac{1}{(1+\sin(x))} \frac{(1-\sin(x))}{(1-\sin(x))}\, dx = \int \frac{(1-\sin(x))}{(1-\sin^2(x))}\, dx.$$

$$I = \int \frac{(1-\sin(x))}{\cos^2(x)}\, dx = \int \frac{dx}{\cos^2(x)} - \int \frac{\sin(x)}{\cos^2(x)}\, dx.$$

Since $\tan'(x) = \frac{1}{\cos^2(x)}$ and $u = \cos(x)$ implies $du = -\sin(x)\, dx$,

$$I = \tan(x) + \int \frac{du}{u^2} = \tan(x) - \frac{1}{u} + c = \tan(x) - \frac{1}{\cos(x)} + c.$$

We conclude that $I = \tan(x) - \sec(x) + c$. $\triangleleft$

Multiplying by 1

Example

Evaluate $I = \int \sec(x)\, dx$.

Solution: This problem can be solved using trigonometric identities for sine and cosine functions only.

$$I = \int \frac{dx}{\cos(x)} = \int \frac{1}{\cos(x)} \, \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)} \, dx$$

$$I = \int \frac{\left(\frac{1}{\cos^2(x)} + \frac{\sin(x)}{\cos^2(x)}\right)}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)} \, dx \quad \text{and} \quad \begin{array}{l} \left(\frac{1}{\cos(x)}\right)' = \frac{\sin(x)}{\cos^2(x)}, \\ \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{1}{\cos^2(x)}. \end{array}$$

$$I = \int \frac{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)'}{\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right)} \, dx \Rightarrow I = \ln\left(\frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)}\right) + c.$$