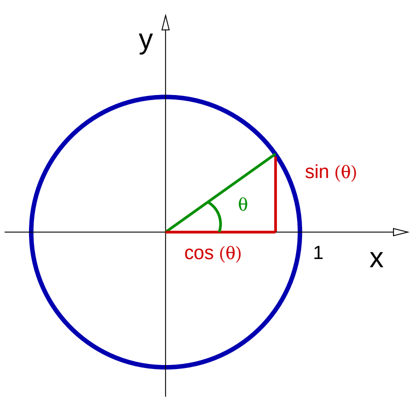
Hyperbolic functions (Sect. 7.7)

- Circular and hyperbolic functions.
- Definitions and identities.
- Derivatives of hyperbolic functions.
- Integrals of hyperbolic functions.

Circular and hyperbolic functions

Remark: Trigonometric functions are also called circular functions.



The circle $x^2 + y^2 = 1$ can be parametrized by the functions

$$x = \cos(\theta),$$
$$y = \sin(\theta).$$

Since these functions satisfy

$$\cos^2(\theta) + \sin^2(\theta) = 1.$$

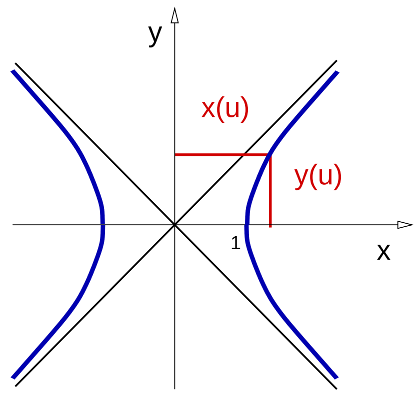
Remark: The parametrization is not unique. Another solution is

$$x = \cos(n\theta), \qquad y = \sin(n\theta), \qquad n \in \mathbb{N}.$$

Circular and hyperbolic functions

Remark:

Hyperbolic functions are a parametrization of a hyperbola.



The hyperbola $x^2 - y^2 = 1$ can be parametrized by the functions

$$x = f(u), \quad y = g(u),$$

satisfying the condition

$$f^2(u) - g^2(u) = 1.$$

Remark: A solution is $x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \; y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$

$$x^2 - y^2 = \frac{1}{4}\Big[h^2 + \frac{1}{h^2} + 2 - h^2 - \frac{1}{h^2} + 2\Big] = 1.$$

Circular and hyperbolic functions

Remarks:

- The hyperbola $x^2 - y^2 = 1$ can be parametrized by

$$x = \frac{1}{2}\Big[h(u) + \frac{1}{h(u)}\Big], \qquad y = \frac{1}{2}\Big[h(u) - \frac{1}{h(u)}\Big],$$

  where h is any non-zero continuous function satisfying

$$\lim_{u\to\infty} h(u) = \infty, \qquad \lim_{u\to-\infty} h(u) = 0, \qquad h(0) = 1.$$

- The hyperbolic trigonometric functions correspond to

$$h(u) = e^u.$$

Definition

The *hyperbolic trigonometric functions* are defined by

$$\cosh(u) = \frac{e^u + e^{-u}}{2}, \qquad \sinh(u) = \frac{e^u - e^{-u}}{2}.$$

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Definitions and identities

Definition

The complete set of *hyperbolic trigonometric functions* is given by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \qquad \sinh(x) = \frac{e^x - e^{-x}}{2},$$

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)},$$

$$\operatorname{csch}(x) = \frac{1}{\sinh(x)}, \qquad \operatorname{sech}(x) = \frac{1}{\cosh(x)}.$$

Remarks:

- These functions satisfy identities similar but not equal to those satisfied by circular trigonometric functions.
- We have seen one of these identities:

$$\cosh^2(x) - \sinh^2(x) = 1.$$

Definitions and identities

Theorem

The following identities hold,

$$\cosh^2(x) - \sinh^2(x) = 1,$$

$$\sinh(2x) = 2\sinh(x)\cosh(x), \quad \cosh(2x) = \cosh^2(x) + \sinh^2(x),$$

$$\cosh^2(x) = \frac{1}{2}\big[1 + \cosh(2x)\big], \quad \sinh^2(x) = \frac{1}{2}\big[-1 + \cosh(2x)\big].$$

Proof: (Only double angle formula for sinh.)

$$\sinh(2x) = \frac{1}{2}\Big[e^{2x} - \frac{1}{e^{2x}}\Big] = \frac{1}{2}\Big[\big(e^x\big)^2 - \Big(\frac{1}{e^x}\Big)^2\Big].$$

Recalling the formula $a^2 - b^2 = (a+b)(a-b)$,

$$\sinh(2x) = \frac{2}{4}\Big[e^x + \frac{1}{e^x}\Big]\Big[e^x - \frac{1}{e^x}\Big] = 2\cosh(x)\sinh(x).$$

□

Definitions and identities

Example

Compute both $\cosh(\ln(7))$ and $\sinh(2\ln(3))$.

Solution:

$$\cosh(\ln(7)) = \frac{1}{2}\Big[e^{\ln(7)} + \frac{1}{e^{\ln(7)}}\Big] = \frac{1}{2}\Big[7 + \frac{1}{7}\Big] = \frac{1}{2}\,\frac{50}{7}.$$

We conclude that $\cosh(\ln(7)) = \dfrac{25}{7}$.

$$\sinh(2\ln(3)) = \frac{1}{2}\Big[e^{2\ln(3)} - \frac{1}{e^{2\ln(3)}}\Big] = \frac{1}{2}\Big[e^{\ln(9)} - \frac{1}{e^{\ln(9)}}\Big]$$

$$\sinh(2\ln(3)) = \frac{1}{2}\Big[9 - \frac{1}{9}\Big] = \frac{1}{2}\,\frac{80}{9} \quad \Rightarrow \quad \sinh(2\ln(3)) = \frac{40}{9}. \lhd$$

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Derivatives of hyperbolic functions

Theorem

The following equations hold,

$$\sinh'(x) = \cosh(x) \qquad \cosh'(x) = \sinh(x)$$

$$\tanh'(x) = \frac{1}{\cosh^2(x)} \qquad \coth'(x) = -\frac{1}{\sinh^2(x)}$$

$$\operatorname{sech}'(x) = -\frac{\sinh(x)}{\cosh^2(x)} \qquad \operatorname{csch}'(x) = -\frac{\cosh(x)}{\sinh^2(x)}.$$

Proof: (Only for sinh.)

$$\sinh'(x) = \frac{1}{2}\big(e^x - e^{-x}\big)' = \frac{1}{2}\big(e^x - e^{-x}(-1)\big)$$

$$\sinh'(u) = \frac{1}{2}\big(e^x + e^{-x}\big) \quad \Rightarrow \quad \sinh'(x) = \cosh(x).$$

□

Derivatives of hyperbolic functions

Example

Compute the derivative of the function $y(x) = e^{\tanh(3x)}$.

Solution:

$$y'(x) = e^{\tanh(3x)} \tanh'(3x)\, 3.$$

We only need to remember the first two formulas in the Theorem above, since

$$\tanh'(x) = \left(\frac{\sinh(x)}{\cosh(x)}\right)' = \frac{\sinh'(x)\cosh(x) - \sinh(x)\cosh'(x)}{\cosh^2(x)}$$

$$\tanh'(x) = \frac{\cosh^2(x) - \sinh^2(x)}{\cosh^2(x)} = \frac{1}{\cosh^2(x)}.$$

We conclude that $y'(x) = \dfrac{3e^{\tanh(3x)}}{\cosh^2(3x)}$. $\triangleleft$

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Integrals of hyperbolic functions

Theorem

For every real constant c the following expressions hold,

$$\int \sinh(x)\,dx = \cosh(x) + c, \qquad \int \cosh(x)\,dx = \sinh(x) + c,$$

$$\int \operatorname{sech}^2(x)\,dx = \tanh(x) + c, \quad \int \operatorname{csch}^2(x)\,dx = -\coth(x) + c,$$

Proof: The derivative of each right-hand side above is the integrand in each left-hand side □

Remark: There are many other integration formulas, but the ones above are the most frequently used.

Integrals of hyperbolic functions

Example

Evaluate $I = \displaystyle\int 6\cosh(3x - \ln(2))\,dx$.

Solution: We try the substitution $u = 3x - \ln(2)$, then $du = 3\,dx$.

$$I = \int 6\cosh(u)\,\frac{du}{3} = 2\int \cosh(u)\,du = 2\sinh(u) + c.$$

We conclude that $I = 2\sinh(3x - \ln(2)) + c$. ◁

Remark: If needed, one can rewrite the sinh above as

$$\sinh(3x - \ln(2)) = \frac{1}{2}\big(e^{3x-\ln(2)} - e^{-3x+\ln(2)}\big)$$

$$\sinh(3x - \ln(2)) = \frac{1}{2}\Big(\frac{e^{3x}}{e^{\ln(2)}} - e^{-3x}\,e^{\ln(2)}\Big) = \frac{e^{3x}}{4} - e^{-3x}.$$

Integrals of hyperbolic functions

Example

Evaluate the integral $I = \int 8x \, \frac{\sinh(3x^2)}{\cosh^3(3x^2)} \, dx$.

Solution: Recall that $\cosh'(x) = \sinh(x)$. We then try the substitution

$$u = \cosh(3x^2), \qquad du = \sinh(3x^2)\, 6x \, dx.$$

$$I = \int 8 \, \frac{1}{u^3} \, \frac{du}{6} = \frac{4}{3} \int u^{-3} \, du = \frac{4}{3} \frac{u^{-2}}{(-2)} + c = -\frac{2}{3} \, \frac{1}{u^2} + c$$

If we substitute back $u = \cosh(3x^2)$, we obtain

$$I = -\frac{2}{3} \, \frac{1}{\cosh^2(3x^2)} + c.$$

◁