Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

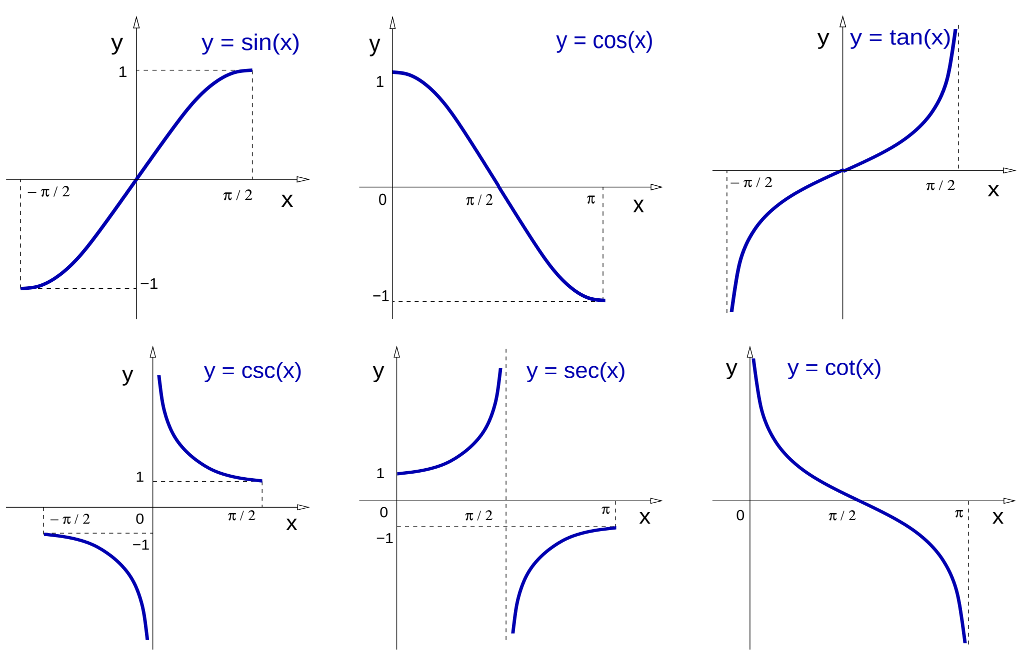
- Review: Definitions and properties.
- Derivatives.
- Integrals.

Last class: Definitions and properties.

- Domains restrictions and inverse trigs.
- Evaluating inverse trigs at simple values.
- Few identities for inverse trigs.

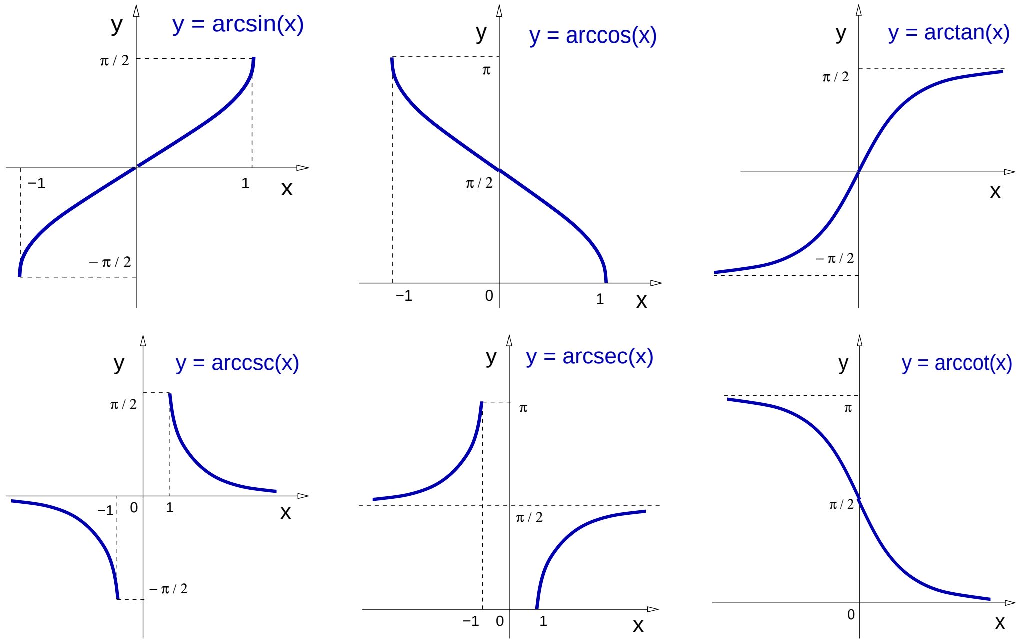
Review: Definitions and properties

Remark: On certain domains the trigonometric functions are invertible.



Review: Definitions and properties

Remark: The graph of the inverse function is a reflection of the original function graph about the $y = x$ axis.



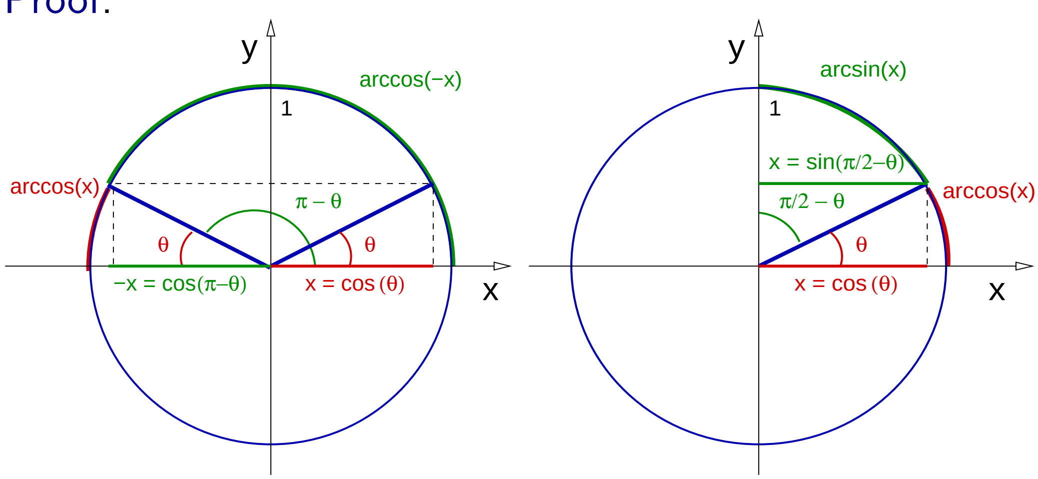
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ the following identities hold,

$$\arccos(x) + \arccos(-x) = \pi, \qquad \arccos(x) + \arcsin(x) = \frac{\pi}{2}.$$

Proof:



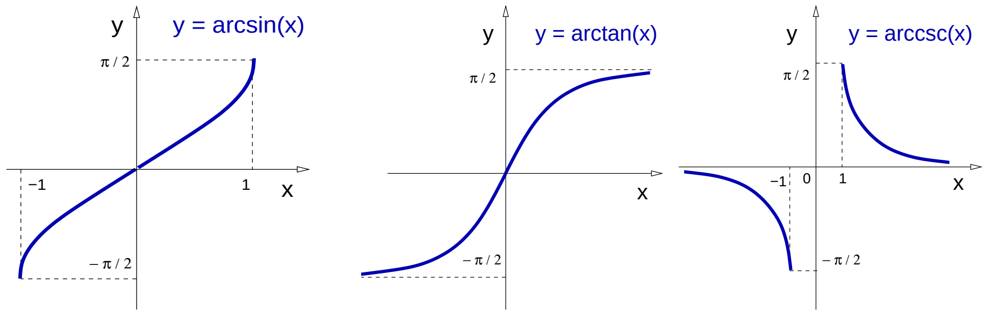
Review: Definitions and properties

Theorem

For all $x \in [-1, 1]$ the following identities hold,

$$\arcsin(-x) = -\arcsin(x),$$
$$\arctan(-x) = -\arctan(x),$$
$$\operatorname{arccsc}(-x) = -\operatorname{arccsc}(x).$$

Proof:



Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

- Review: Definitions and properties.
- **Derivatives.**
- Integrals.

Derivatives of inverse trigonometric functions

Remark: Derivatives inverse functions can be computed with

$$(f^{-1})'(x) = \frac{1}{f'\big(f^{-1}(x)\big)}.$$

Theorem

The derivative of arcsin *is given by* $\arcsin'(x) = \dfrac{1}{\sqrt{1-x^2}}$.

Proof: For $x \in [-1, 1]$ holds

$$\arcsin'(x) = \frac{1}{\sin'\big(\arcsin(x)\big)} = \frac{1}{\cos\big(\arcsin(x)\big)}$$

For $x \in [-1, 1]$ we get $\arcsin(x) = y \in \left[\frac{\pi}{2}, \frac{\pi}{2}\right]$, and the cosine is positive in that interval, then $\cos(y) = +\sqrt{1 - \sin^2(y)}$, hence

$$\arcsin'(x) = \frac{1}{\sqrt{1 - \sin^2\big(\arcsin(x)\big)}} \quad \Rightarrow \quad \arcsin'(x) = \frac{1}{\sqrt{1-x^2}}. \quad \square$$

Derivatives of inverse trigonometric functions

Theorem

The derivative of inverse trigonometric functions are:

$$\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \qquad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}, \qquad |x| \leqslant 1,$$

$$\arctan'(x) = \frac{1}{1+x^2}, \qquad \text{arccot}'(x) = -\frac{1}{1+x^2}, \qquad x \in \mathbb{R},$$

$$\text{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2-1}}, \qquad \text{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2-1}}, \qquad |x| \geqslant 1.$$

Proof: $\arctan'(x) = \dfrac{1}{\tan'\big(\arctan(x)\big)}$, $\tan'(y) = \dfrac{\cos^2(y) + \sin^2(y)}{\cos^2(y)}$

$$\tan'(y) = 1 + \tan^2(y), \;\; y = \arctan(x), \quad \Rightarrow \quad \arctan'(x) = \frac{1}{1+x^2}.$$

Derivatives of inverse trigonometric functions

Proof: $\operatorname{arcsec}'(x) = \dfrac{1}{\sec'\big(\operatorname{arcsec}(x)\big)}$, for $|x| \geqslant 1$.

Then $y = \operatorname{arcsec}(x)$ satisfies $y \in [0, \pi] - \{\pi/2\}$. Recall,

$$\sec'(y) = \Big(\frac{1}{\cos(y)}\Big)' = \frac{\sin(y)}{\cos^2(y)}, \quad \sin(y) = +\sqrt{1 - \cos^2(y)},$$

$$\sec'(y) = \frac{\sqrt{1 - \cos^2(y)}}{\cos^2(y)} = \frac{1}{|\cos(y)|} \frac{\sqrt{1 - \cos^2(y)}}{|\cos(y)|},$$

$$\sec'(y) = \frac{1}{|\cos(y)|} \sqrt{\frac{1}{\cos^2(y)} - 1} = |\sec(y)| \sqrt{\sec^2(y) - 1}.$$

We conclude: $\operatorname{arcsec}'(x) = \dfrac{1}{|x| \sqrt{x^2 - 1}}$. □

Derivatives of inverse trigonometric functions

Example

Compute the derivative of $y(x) = \operatorname{arcsec}(3x + 7)$.

Solution: Recall the main formula: $\operatorname{arcsec}'(u) = \dfrac{1}{|u|\sqrt{u^2 - 1}}$.

Then, chain rule implies, $y'(x) = \dfrac{3}{|3x + 7| \sqrt{(3x + 7)^2 - 1}}$. ◁

Example

Compute the derivative of $y(x) = \arctan(4 \ln(x))$.

Solution: Recall the main formula: $\arctan'(u) = \dfrac{1}{1 + u^2}$.

Therefore, chain rule implies,

$$y'(x) = \frac{1}{\big[1 + \big(4 \ln(x)\big)^2\big]} \frac{4}{x} \quad \Rightarrow \quad y' = \frac{4}{x\big[1 + 16 \ln^2(x)\big]}. \quad \triangleleft$$

Inverse trigonometric functions (Sect. 7.6)

Today: Derivatives and integrals.

- Review: Definitions and properties.
- Derivatives.
- **Integrals.**

Integrals of inverse trigonometric functions

Remark: The formulas for the derivatives of inverse trigonometric functions imply the integration formulas.

Theorem

For any constant $a \neq 0$ holds,

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin\Big(\frac{x}{a}\Big) + c, \qquad |x| < a,$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\arctan\Big(\frac{x}{a}\Big) + c, \qquad x \in \mathbb{R},$$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\,\mathrm{arcsec}\Big(\Big|\frac{x}{a}\Big|\Big) + c, \qquad |x| > a > 0.$$

Proof: (For arcsine only.) $y(x) = \arcsin\Big(\dfrac{x}{a}\Big) + c$, then

$$y'(x) = \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}}\,\frac{1}{a} = \frac{|a|}{\sqrt{a^2 - x^2}}\,\frac{1}{a} \;\Rightarrow\; y'(x) = \frac{1}{\sqrt{a^2 - x^2}} \qquad \square$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{\sqrt{3 - 4(x-1)^2}}\, dx.$

Solution: Substitute: $u = 2(x-1)$, then $du = 2\, dx$,

$$I = \int \frac{6}{\sqrt{3-u^2}} \frac{du}{2} = 3 \int \frac{du}{\sqrt{3-u^2}}.$$

Recall: $\displaystyle\int \frac{dx}{\sqrt{a^2-u^2}} = \arcsin\Big(\frac{u}{a}\Big) + c.$ Then, for $a = \sqrt{3}$,

$$I = 3\, \arcsin\Big(\frac{u}{\sqrt{3}}\Big) + c \quad \Rightarrow \quad I = 3\, \arcsin\Big(\frac{2(x-1)}{\sqrt{3}}\Big) + c. \quad \triangleleft$$

Integrals of inverse trigonometric functions

Example

Evaluate $I = \displaystyle\int \frac{6}{t\big[\ln^2(t) + \ln(t^4) + 8\big]}\, dt.$

Solution: Recall: $\ln(t^4) = 4\ln(t)$, Try to complete the square.

$$I = \int \frac{6}{t\big[\ln^2(t) + 4\ln(t) + 8\big]}\, dt,$$

$$I = \int \frac{6}{t\big[\ln^2(t) + 2(2\ln(t)) + 4 - 4 + 8\big]}\, dt$$

$$I = \int \frac{6}{t\big[\big(\ln(t) + 2\big)^2 + 4\big]}\, dt$$

This looks like the derivative of the arctangent.

Integrals of inverse trigonometric functions

Example

Evaluate $I = \int \frac{6}{t\left[\ln^2(t) + \ln(t^4) + 8\right]}\, dt.$

Solution: Recall: $I = \int \frac{6}{t\left[\left(\ln(t) + 2\right)^2 + 4\right]}\, dt.$

Substitute: $u = \ln(t) + 2$, then $du = \frac{1}{t}\, dt$,

$$I = \int \frac{6}{4 + u^2}\, du = 6 \int \frac{du}{2^2 + u^2} = 6\, \frac{1}{2} \arctan\left(\frac{u}{2}\right) + c.$$

$I = 3 \arctan\left(\frac{1}{2}(\ln(t)+2)\right) + c \quad \Rightarrow \quad I = 3 \arctan\left(\ln(\sqrt{t}) + 1\right) + c.$

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