Solving differential equations (Sect. 7.4)

Today: Applications.

- Review: Overview of differential equations.
- Population growth.
- Radioactive decay.
- Newton's Law of Cooling.

Previous class:

- Overview of differential equations.
- Exponential growth.
- Separable differential equations.

Review: Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Recall:

(a) All solutions y to the exponential growth equation $y'(x) = k\, y(x)$, with constant k, are given by the exponentials

$$y(x) = y_0\, e^{kx},$$

where $y(0) = y_0$.

(b) All solutions y to the separable equation $h(y)\, y'(x) = g(x)$, with functions h, g, are given in implicit form,

$$H(y) = G(x) + c,$$

where $H' = h$ and $g' = g$.

Review: Overview of differential equations.

Example

Find all solutions y to the equation $y'(x) = \dfrac{e^{2x-y}}{e^{x+y}}$.

Solution: Rewrite the differential equation,

$$y' = \frac{e^{2x}\, e^{-y}}{e^{x}\, e^{y}} = e^{2x}\, e^{-y} \frac{1}{e^{x}} \frac{1}{e^{y}} = e^{2x}\, e^{-x}\, e^{-y}\, e^{-y}.$$

$$y' = e^{x}\, e^{-2y} = \frac{e^{x}}{e^{2y}} \quad \Rightarrow \quad e^{2y}\, y' = e^{x}.$$

Hence, the equation is separable. We integrate on both sides,

$$\int e^{2y(x)}\, y'(x)\, dx = \int e^{x}\, dx.$$

Review: Overview of differential equations.

Example

Find all solutions y to the equation $y'(x) = \dfrac{e^{2x-y}}{e^{x+y}}$.

Solution: Recall: $\displaystyle\int e^{2y(x)}\, y'(x)\, dx = \int e^{x}\, dx$.

The usual substitution $u = y(x)$, and then $du = y'(x)\, dx$,

$$\int e^{2u}\, du = \int e^{x}\, dx \quad \Rightarrow \quad \frac{1}{2}\, e^{2u} = e^{x} + c.$$

We now substitute back $u = y(x)$,

$$e^{2y(x)} = 2\,(e^{x} + c) \quad \Rightarrow \quad 2y(x) = \ln\big(2\,(e^{x} + c)\big)$$

We conclude that $y(x) = \dfrac{1}{2}\, \ln\big(2\,(e^{x} + c)\big)$. $\triangleleft$

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Population growth

Example

Assume the world population growth is described by $y(t) = y_0\, e^{k(t-t_0)}$, with t measured in years.

(a) If in $1960 - 1961$ the population increased by 2%, find k.

(b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above.

Solution: (a) $y(1961) = \Big(1 + \dfrac{2}{100}\Big)\, y(1960)$,

$$y_0\, e^{k(1961-t_0)} = \frac{102}{100}\, y_0\, e^{k(1960-t_0)}$$

$$e^{k1961}\, e^{-kt_0} = 1.02\, e^{k1960}\, e^{-kt_0} \;\Rightarrow\; e^{k(1961-1960)} = 1.02.$$

$e^{k} = 1.02 \;\Rightarrow\; k = \ln(1.02) \simeq 0.02.$ Hence $y(t) = y_0\, e^{(0.02)(t-t_0)}$.

Population growth

Example

Assume the world population growth is described by $y(t) = y_0\, e^{k(t-t_0)}$, with t measured in years.

(a) If in $1960 - 1961$ the population increased by 2%, find k.

(b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above.

Solution: Recall: $y(t) = y_0\, e^{(0.02)(t-t_0)}$.

(b) If y represents billions of people,

$$3 = y(t_0) = y_0\, e^{(0.02)(t_0-t_0)} \;\Rightarrow\; y_0 = 3 \;\Rightarrow\; y(t) = 3\, e^{(0.02)(t-1960)}.$$

We only need to evaluate $y(2012) = 3\, e^{(0.02)52} = 8.5$ billions. $\triangleleft$

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Radioactive decay

Remarks:

- Some atoms can spontaneously break into smaller atoms.
- This process is called radioactive decay.
- It can be seen that the concentration y of a radioactive substance in time t follows the law,

$$y'(t) = -k\, y(t), \qquad k > 0.$$

- We know the solution is

$$y(t) = y_0\, e^{-kt}, \qquad y(0) = y_0.$$

- The *half-life* of the material is the τ such that $y(\tau) = \dfrac{y_0}{2}$.

$$\frac{y_0}{2} = y_0\, e^{-k\tau} \;\Rightarrow\; -k\tau = \ln\Big(\frac{1}{2}\Big) \;\Rightarrow\; \tau = \frac{\ln(2)}{k}.$$

Radioactive decay

Example

The half-life of a radioactive material is $\tau = 5730$ years. If a material sample contains 14% of the original amount, find the date the material sample was created.

Solution: Let us fix the time of the original amount at $t = 0$, and denote the present time by t_1. Also denote $y(t)$ the material amount at time t.

$$y(t) = y_0\, e^{-kt} \quad\Rightarrow\quad y_0\, e^{-kt_1} = y(t_1) = \frac{14}{100}\, y(0) = \frac{14}{100}\, y_0.$$

$$y_0\, e^{-kt_1} = \frac{14}{100}\, y_0 \;\Rightarrow\; -kt_1 = \ln\Big(\frac{14}{100}\Big) \;\Rightarrow\; t_1 = \frac{1}{k}\ln\Big(\frac{100}{14}\Big).$$

Recall $\tau = \ln(2)/k$ and $\tau = 5730$ years. So $1/k = 5730/\ln(2)$,

We obtain $t_1 = [5730/\ln(2)]\,\ln\Big(\dfrac{100}{14}\Big)$, hence $t_1 = 16{,}253$ years.$\lhd$

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Newton's Law of Cooling.

Remarks:

- The temperature difference $\Delta T = T - T_0$ between the temperature of an object, T, and the constant temperature of the surrounding medium where it is placed, T_s, evolves in time t following the equation

$$(\Delta T)' = -k(\Delta T), \quad T(0) = T_0, \quad k > 0.$$

- The solution is $(\Delta T)(t) = (\Delta T)_0 \, e^{-kt}$, that is,

$$(T - T_s)(t) = (T_0 - T_s) \, e^{-kt}$$

$$T(t) = (T_0 - T_s) \, e^{-kt} + T_s.$$

- The constant k depends on the material and the surroundings.

Newton's Law of Cooling.

Example

A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C? while

Solution: We know that $T(t) = (T_0 - T_s)\, e^{-kt} + T_s$, and also

$$T_0 = 45, \qquad T_s = 5, \qquad T(2) = 25.$$

Find t_1 such that $T(t_1) = 15$. First we find k,

$$T(t) = (45 - 5)\, e^{-kt} + 5 \quad \Rightarrow \quad T(t) = 40\, e^{-kt} + 5.$$

$$20 = T(2) = 40\, e^{-2k} \;\Rightarrow\; \ln(1/2) = -2k \;\Rightarrow\; k = \frac{1}{2}\ln(2).$$

$$T(t) = 40\, e^{-t \ln(\sqrt{2})} + 5 \Rightarrow 10 = 40\, e^{-t_1 \ln(\sqrt{2})} \Rightarrow t_1 = 4. \quad \triangleleft$$