

Review: Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Recall:

(a) All solutions y to the exponential growth equation

y'(x) = k y(x), with constant k, are given by the exponentials

 $y(x)=y_0\,e^{kx},$

where $y(0) = y_0$.

(b) All solutions y to the separable equation h(y) y'(x) = g(x), with functions h, g, are given in implicit form,

$$H(y) = G(x) + c,$$

where H' = h and g' = g.

Review: Overview of differential equations.

Example

Find all solutions y to the equation $y'(x) = \frac{e^{2x-y}}{e^{x+y}}$.

Solution: Rewrite the differential equation,

$$y' = rac{e^{2x} e^{-y}}{e^{x} e^{y}} = e^{2x} e^{-y} rac{1}{e^{x}} rac{1}{e^{y}} = e^{2x} e^{-x} e^{-y} e^{-y}.$$

$$y' = e^x e^{-2y} = \frac{e^x}{e^{2y}} \quad \Rightarrow \quad e^{2y} y' = e^x.$$

Hence, the equation is separable. We integrate on both sides,

$$\int e^{2y(x)} y'(x) \, dx = \int e^x \, dx.$$

Review: Overview of differential equations.

Example

Find all solutions y to the equation $y'(x) = \frac{e^{2x-y}}{e^{x+y}}$.

Solution: Recall: $\int e^{2y(x)} y'(x) dx = \int e^x dx.$

The usual substitution u = y(x), and then du = y'(x) dx,

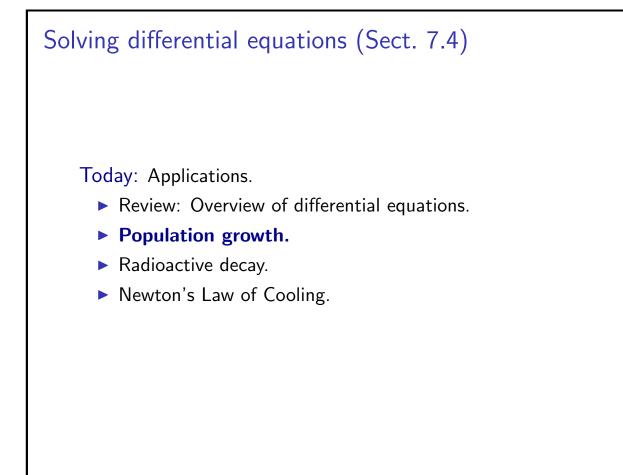
$$\int e^{2u} du = \int e^x dx \quad \Rightarrow \quad \frac{1}{2} e^{2u} = e^x + c.$$

We now substitute back u = y(x),

$$e^{2y(x)} = 2(e^x + c) \quad \Rightarrow \quad 2y(x) = \ln(2(e^x + c))$$

We conclude that $y(x) = \frac{1}{2} \ln(2(e^x + c))$.

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Population growth

Example

Assume the world population growth is described by $y(t) = y_0 e^{k(t-t_0)}$, with t measured in years.

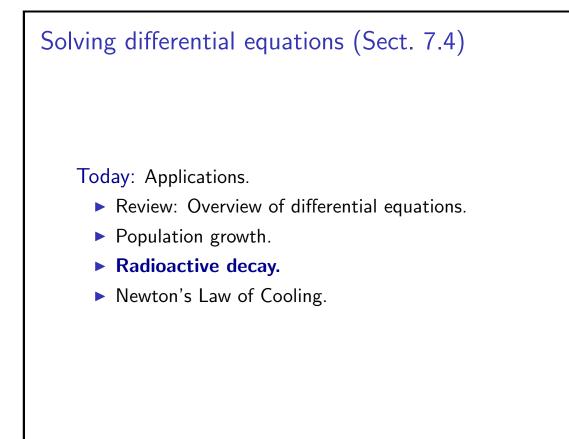
- (a) If in 1960 1961 the population increased by 2%, find k.
- (b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above.

Solution: (a)
$$y(1961) = \left(1 + \frac{2}{100}\right)y(1960),$$

$$y_0 e^{k(1961-t_0)} = \frac{102}{100} y_0 e^{k(1960-t_0)}$$
$$e^{k1961} e^{-kt_0} = 1.02 e^{k1960} e^{-kt_0} \implies e^{k(1961-1960)} = 1.02.$$
$$e^k = 1.02 \implies k = \ln(1.02) \simeq 0.02. \text{ Hence } y(t) = y_0 e^{(0.02)(t-t_0)}.$$

Population growth Example Assume the world population growth is described by $y(t) = y_0 e^{k(t-t_0)}$, with t measured in years. (a) If in 1960 – 1961 the population increased by 2%, find k. (b) If the population in $t_0 = 1960$ was 3 billion people, find the actual population predicted by the law above. Solution: Recall: $y(t) = y_0 e^{(0.02)(t-t_0)}$. (b) If y represents billions of people, $3 = y(t_0) = y_0 e^{(0.02)(t_0-t_0)} \Rightarrow y_0 = 3 \Rightarrow y(t) = 3 e^{(0.02)(t-1960)}$.

We only need to evaluate $y(2012) = 3 e^{(0.02)52} = 8.5$ billions.



Radioactive decay

Remarks:

- Some atoms can spontaneously break into smaller atoms.
- This process is called radioactive decay.
- It can be seen that the concentration y of a radioactive substance in time t follows the law,

$$y'(t) = -k y(t), \qquad k > 0.$$

We know the solution is

$$y(t) = y_0 e^{-kt}, \qquad y(0) = y_0$$

• The *half-life* of the material is the τ such that $y(\tau) = \frac{y_0}{2}$.

$$\frac{y_0}{2} = y_0 e^{-k\tau} \Rightarrow -k\tau = \ln\left(\frac{1}{2}\right) \Rightarrow \tau = \frac{\ln(2)}{k}$$

Radioactive decay

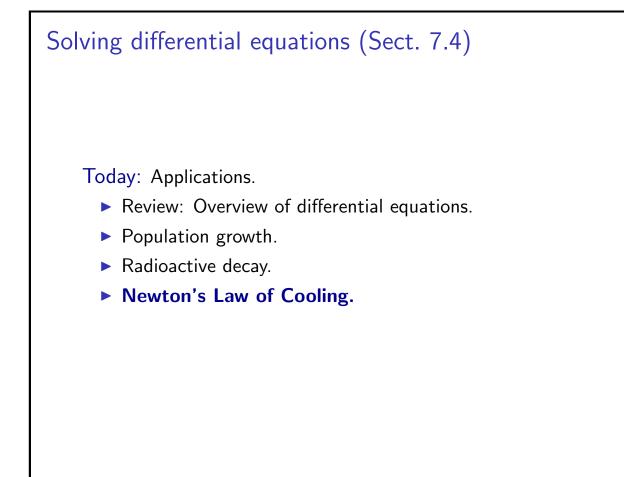
Example

The half-life of a radioactive material is $\tau = 5730$ years. If a material sample contains 14% of the original amount, find the date the material sample was created.

Solution: Let us fix the time of the original amount at t = 0, and denote the present time by t_1 . Also denote y(t) the material amount at time t.

$$y(t) = y_0 e^{-kt} \quad \Rightarrow \quad y_0 e^{-kt_1} = y(t_1) = \frac{14}{100} y(0) = \frac{14}{100} y_0.$$
$$y_0 e^{-kt_1} = \frac{14}{100} y_0 \quad \Rightarrow \quad -kt_1 = \ln\left(\frac{14}{100}\right) \quad \Rightarrow \quad t_1 = \frac{1}{k} \ln\left(\frac{100}{14}\right).$$

Recall $\tau = \ln(2)/k$ and $\tau = 5730$ years. So $1/k = 5730/\ln(2)$, We obtain $t_1 = [5730/\ln(2)] \ln(\frac{100}{14})$, hence $t_1 = 16,253$ years.



Newton's Law of Cooling.

Remarks:

► The temperature difference ΔT = T - T₀ between the temperature of an object, T, and the constant temperature of the surrounding medium where it is placed, T_s, evolves in time t following the equation

$$(\Delta T)' = -k(\Delta T), \quad T(0) = T_0, \quad k > 0.$$

• The solution is $(\Delta T)(t) = (\Delta T)_0 e^{-kt}$, that is,

$$(T-T_s)(t) = (T_0 - T_s) e^{-kt}$$

 $T(t) = (T_0 - T_s) e^{-kt} + T_s.$

▶ The constant *k* depends on the material and the surroundings.

Newton's Law of Cooling.

Example

A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C? while

Solution: We know that $T(t) = (T_0 - T_s) e^{-kt} + T_s$, and also

$$T_0 = 45, \qquad T_s = 5, \qquad T(2) = 25.$$

Find t_1 such that $T(t_1) = 15$. First we find k,

$$T(t) = (45-5) e^{-kt} + 5 \quad \Rightarrow \quad T(t) = 40 e^{-kt} + 5.$$

$$20 = T(2) = 40 e^{-2k} \Rightarrow \ln(1/2) = -2k \Rightarrow k = \frac{1}{2}\ln(2).$$

$$T(t) = 40 e^{-t \ln(\sqrt{2})} + 5 \Rightarrow 10 = 40 e^{-t_1 \ln(\sqrt{2})} \Rightarrow t_1 = 4. \quad \lhd$$