Review for Midterm Exam 1.

- 5 or 6 problems.
- No multiple choice questions.
- No notes, no books, no calculators.
- Problems similar to webwork.
- Midterm Exam 1 covers:
  - Volumes using cross-sections (6.1).
  - Arc-length of curves on the plane (6.3).
  - Work and fluid forces (6.5).
  - The inverse function (7.1).
  - The natural logarithm (7.2).
  - The exponential function (7.3).

Volumes using cross-sections (6.1)

Example

Find the volume of the region obtained by rotation the curve $x(y) = \tan(\pi y/8)$ for $y \in [0, 2]$ about the y-axis.

Solution:

To graph the function

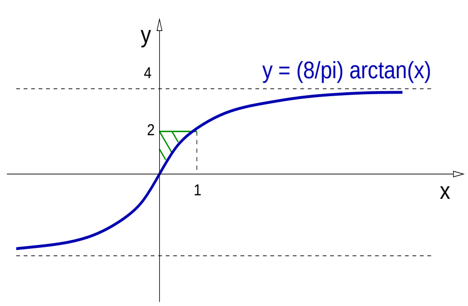
$$x = \tan(\pi y/8), \;\; y \in [0, 2],$$

one can graph

$$y = (8/\pi) \arctan(x).$$

Notice that

$$y \in [0, 2] \Rightarrow x \in [0, 1].$$



Therefore, $V = \pi \displaystyle\int_0^2 \left[x(y)\right]^2 dy = \pi \int_0^2 \left[\tan\left(\frac{\pi y}{8}\right)\right]^2 dy.$

Volumes using cross-sections (6.1)

Example

Find the volume of the region obtained by rotation the curve $x(y) = \tan(\pi y/8)$ for $y \in [0, 2]$ about the y-axis.

Solution: Recall: $V = \pi \int_0^2 \tan^2\Big(\frac{\pi y}{8}\Big)\, dy.$

Introduce the substitution $u = \pi y/8$, so $du = (\pi/8)\, dy$,

$$V = \pi\,\frac{8}{\pi}\int_0^{\pi/4} \tan^2(u)\, du = 8\int_0^{\pi/4} \frac{[1-\cos^2(u)]}{\cos^2(u)}\, du$$

$$V = 8\int_0^{\pi/4}\Big[\frac{1}{\cos^2(u)} - 1\Big]\, du = 8\int_0^{\pi/4}\big[\tan'(u) - 1\big]\, du.$$

$$V = 8\big[\tan(u) - u\big]\Big|_0^{\pi/4} \quad\Rightarrow\quad V = 8\Big(1 - \frac{\pi}{4}\Big).$$

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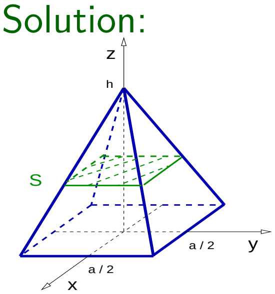
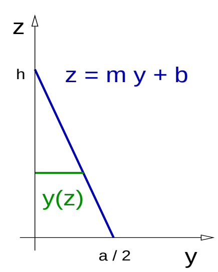
Volumes integrating cross-sections: General case.

Example

Find the volume of a pyramid with square base side a and height h.

Solution:





$$A(z) = \big[2y(z)\big]^2$$

We must find and invert

$$z(y) = my + b.$$

$$h = z(0) = b, \quad 0 = z(a/2) = m\frac{a}{2} + h \quad\Rightarrow\quad m = -\frac{2h}{a}.$$

$$z(y) = -\frac{2h}{a}\,y + h \quad\Rightarrow\quad y(z) = -\frac{a}{2h}\,(z - h).$$

$$V = \int_0^h \Big[-2\frac{a}{2h}\,(z-h)\Big]^2 dz = \frac{a^2}{h^2}\Big[\frac{(z-h)^3}{3}\Big|_0^h\Big] \Rightarrow V = \frac{1}{3}a^2 h.$$

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Arc-length of curves on the plane (6.3)

Example

Find the arc-length of the function $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$, for $x \in [1, 3]$.

Solution: Recall: $L = \displaystyle\int_{x_0}^{x_1} \sqrt{1 + \big[y'(x)\big]^2}\, dx$. Find y',

$$y'(x) = x^2 - \frac{1}{4x^2} \quad \Rightarrow \quad 1 + \big[y'(x)\big]^2 = 1 + x^4 + \frac{1}{16x^4} - \frac{1}{2},$$

$$1 + \big[y'(x)\big]^2 = x^4 + \frac{1}{16x^4} + \frac{1}{2} = \Big(x^2 + \frac{1}{4x^2}\Big)^2.$$

$$L = \int_1^3 \Big(x^2 + \frac{1}{4x^2}\Big)\, dx = \Big(\frac{x^3}{3} - \frac{1}{4x}\Big)\Big|_1^3 = 9 - \frac{1}{12} - \frac{1}{3} + \frac{1}{4}.$$

We conclude that $L = 9 - 1/6$. ◁

The main length formula

Example

Find the arc-length of the curve $y = x^{3/2}$, for $x \in [0, 4]$.

Solution: Recall: $L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$. We start with

$$f(x) = x^{3/2} \quad \Rightarrow \quad f'(x) = \frac{3}{2}\, x^{1/2} \quad \Rightarrow \quad [f'(x)]^2 = \frac{9}{4}\, x.$$

$$L = \int_0^4 \sqrt{1 + \frac{9}{4} x}\, dx, \quad u = 1 + \frac{9}{4}\, x, \quad du = \frac{9}{4}\, dx.$$

$$L = \int_1^{10} \frac{4}{9} \sqrt{u}\, du = \frac{4}{9}\,\frac{2}{3} \left(u^{3/2} \Big|_1^{10} \right).$$

We conclude that $L = \dfrac{8}{27}(10^{3/2} - 1)$. $\triangleleft$

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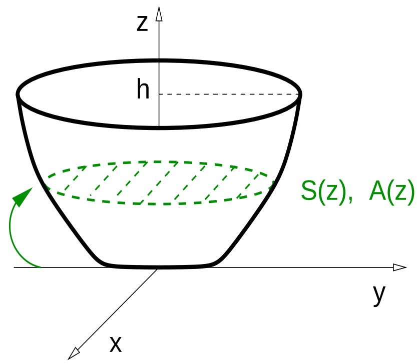
Work and fluid forces: Pumping liquids

Proof: (a) Show: $W = \int_0^{h_1} g\,\delta\,A(z)\,z\,dz$.



The amount of liquid that can be placed at cross-section $S(z)$ is

$$M = \delta\,A(z)\,dz.$$

The force that must be done to lift that amount of liquid is

$$F = g\,[\delta\,A(z)\,dz].$$

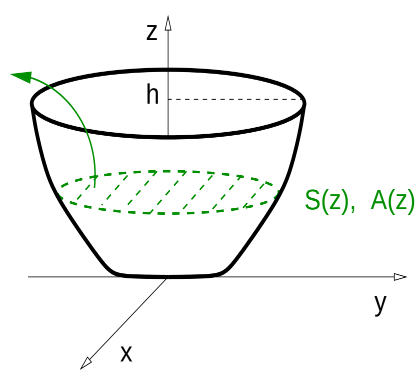
The work done to lift that liquid to height z from $z = 0$ is

$$W(z) = z\,g\,[\delta\,A(z)\,dz].$$

The work to fill in the container up to h_1 is $W = \int_0^{h_1} g\,\delta\,A(z)\,z\,dz$.

Work and fluid forces: Pumping liquids

Proof: (b) Show: $W = \int_0^{h_1} g\,\delta\,A(z)\,(h - z)\,dz$.



The force that must be done to lift the liquid in $S(z)$ is

$$F = g\,[\delta\,A(z)\,dz].$$

The work done to lift that liquid from a height z to h is

$$W(z) = (h - z)\,g\,[\delta\,A(z)\,dz].$$

The work to empty the container initially filled up to h_1 is

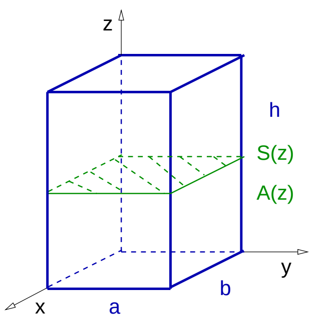
$$W = \int_0^{h_1} g\,\delta\,A(z)\,(h - z)\,dz.$$

Work and fluid forces: Pumping liquids

Example

A rectangular container with sides a, b, and height h, is filled with water. Find the work needed to empty the container if the water is pumped from the top of the tank. Recall the water density is $\delta = 1000\ Kg/m^3$, and the gravity acceleration is $g = 10\ m/s^2$.

Solution:



The force is the water weight:

$$F = g\,[\delta\, A(z)\, dz] = g\delta(ab)\, dz$$

The work done to lift that liquid from a height z to h is

$$W(z) = g\delta(ab)(h - z)\, dz.$$

To empty the container: $W = g\delta(ab)\int_0^h (h - z)\, dz = g\delta\,(ab)\dfrac{h^2}{2}.$

Work and fluid forces: Springs

Remark: The force of a spring, $F(x) = k\,x$ is called *Hooke's Law*.

Example

Find the minimum work needed to compress a spring with constant $k = 3\ N/m$ a distance of $d\ m$ from the spring rest position.

Solution: The spring force is $F(x) = kx$, then

$$W = \int_0^d kx\, dx = k\frac{x^2}{2}\Big|_0^d \quad \Rightarrow \quad W = \frac{kd^2}{2}. \qquad \triangleleft\ .$$

Example

If a force of 60 N stretches a spring 3 m from its rest position, how much work does it take to stretch it 4 m from its rest position?

Solution: From Hooke's Law we know that $60\ N = k\ (3)\ m$, that is, $k = 20\ N/m$. The previous problem implies $W = kd^2/2$, that is,

$$W = 20\,\frac{N}{m}\,\frac{4^2}{2}\ m^2 \quad \Rightarrow \quad W = 160\ J. \qquad \triangleleft$$

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The inverse function (7.1).

Example

Find the inverse of $f(x) = 8(x-2)^2 + 3$ for $x \geqslant 2$.

Solution: We call $y = f(x)$, and we find $x(y)$.

$$y = 8(x-2)^2 + 3 \quad \Rightarrow \quad (x-2)^2 = \frac{1}{8}(y-3)$$

$$x - 2 = \sqrt{\frac{1}{8}(y-3)} \quad \Rightarrow \quad x = 2 + \sqrt{\frac{1}{8}(y-3)}.$$

Example

Given $f(x) = 2x^3 + 3x^2 + 3$ for $x \geqslant 0$, find $\dfrac{df^{-1}}{dx}$ at $x = 8 = f(1)$.

Solution: Recall: $(f^{-1})'(8) = \dfrac{1}{f'(f^{-1}(8))}$. Since $f^{-1}(8) = 1$,

we need $f'(1)$. Since $f'(x) = 6x^2 + 6x$, we get $f'(1) = 12$.

We obtain $(f^{-1})'(8) = \dfrac{1}{12}$. ◁

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The natural logarithm (7.2)

Example

Simplify $f(x) = \ln\Big(\dfrac{\sin^5(2t)}{7}\Big)$, and find the derivatives of $g(x) = 3\ln(6\ln(x))$, and $h(x) = \ln\big(\sqrt{25\sin(x)\cos(x)}\big)$.

Solution: First: $f(x) = \ln(\sin^5(2t)) - \ln(7)$,

so we conclude that $f(x) = 5\ln(\sin(2t)) - \ln(7)$.

Second, $g'(x) = 3\,\dfrac{1}{6\ln(x)}\,(6\ln(x))'$, that is, $g'(x) = 3\,\dfrac{1}{\ln(x)}\,\dfrac{1}{x}$.

Sometimes it is better simplify first and derivate later,

$$h(x) = \frac{1}{2}\big[\ln(25) + \ln(\sin(x)) + \ln(\cos(x)\big],$$

$$h'(x) = \frac{1}{2}\Big[\frac{\cos(x)}{\sin(x)} - \frac{\sin(x)}{\cos(x)}\Big].$$

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The natural logarithm (7.2)

Example

Find $I = \int \frac{\sec(x)}{\sqrt{\ln(\sec(x)+\tan(x))}}\,dx$.

Solution: We try the substitution $u = \ln(\sec(x)+\tan(x))$. Recall

$$\sec(x)+\tan(x) = \frac{1}{\cos(x)} + \frac{\sin(x)}{\cos(x)} = \frac{1+\sin(x)}{\cos(x)},$$

$$du = \frac{\cos(x)}{1+\sin(x)}\left[\frac{\cos(x)\cos(x) - (1+\sin(x))(-\sin(x))}{\cos^2(x)}\right] dx$$

$$du = \frac{\cos(x)}{[1+\sin(x)]}\,\frac{[1+\sin(x)]}{\cos^2(x)}\,dx = \frac{1}{\cos(x)}\,dx = \sec(x)\,dx.$$

$$I = \int \frac{du}{u^{1/2}} = 2\,u^{1/2} \Rightarrow I = 2\sqrt{\ln(\sec(x)+\tan(x))}. \qquad \triangleleft$$

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The exponential function (7.3)

Example

Solve for y in terms of x the equation

$$\ln(3y-5)+\ln(2)=4x+\ln(2x).$$

Solution:

$$\ln\Big(\frac{3y-5}{2}\Big)=\ln\big(e^{4x}\big)+\ln(2x)=\ln\big(2x\,e^{4x}\big).$$

$$\frac{3y-5}{2}=2x\,e^{4x}\quad\Rightarrow\quad 3y=4x\,e^{4x}+5$$

$$y=\frac{1}{3}\Big(4x\,e^{4x}+5\Big).\qquad\triangleleft$$

The exponential function (7.3)

Example

Solve the initial value problem

$$y'(x)=5\,e^{5x}\,\sin\big(e^{5x}-2\big),\qquad y\Big(\frac{\ln(2)}{5}\Big)=0.$$

Solution: We need to compute the integral

$$y(x)=\int 5\,e^{5x}\,\sin\big(e^{5x}-2\big)\,dx+c$$

Substitute $u=e^{5x}-2$, then $du=5\,e^{5x}\,dx$, so

$$y(x)=\int\sin(u)\,du+c=-\cos(u)+c$$

So $y(x)=-\cos\big(e^{5x}-2\big)+c$. The initial condition implies

$$0=y\Big(\frac{\ln(2)}{5}\Big)=-\cos\big(e^{\ln(2)}-2\big)+c=-\cos(2-2)+c=-1+c$$

We conclude that $c=1$, so $y(x)=-\cos\big(e^{5x}-2\big)+1$. $\triangleleft$