Solving differential equations (Sect. 7.4)

- Overview of differential equations.
- Exponential growth.
- Separable differential equations.

Next class: Applications.

- Population growth
- Radioactive decay.
- Heat transfer.

Overview of differential equations.

Definition

A *differential equation* is an equation, where the unknown is a function, and both the function and its derivative appear in the equation.

Remark: May be the most famous differential equation is Newton's second law of motion: $ma = f$.

Example

Newton's second law of motion in one space dimension is a differential equation: The unknown is $x(t)$, the particle position as function of time t, and the equation is

$$\frac{d^2x}{dt^2}(t) = \frac{1}{m} f(t, x(t)),$$

with m the particle mass and f the force acting on the particle.

Overview of differential equations.

Example

The following are examples of differential equations:

- Given a constant k find every function y solution of

$$\frac{dy}{dx}(x) = k\, y(x).$$

- Find the function y solution of

$$\frac{dy}{dx}(x) = (1 + y(x))\, x^2, \qquad y(0) = 2.$$

- Find every function y solution of

$$\frac{dy}{dx}(x) = \frac{e^{2x - y(x)}}{e^{x + y(x)}}.$$

Overview of differential equations.

Example

Verify that the functions $y(x) = c\, e^{-2x} - \frac{3}{2}$, for every $c \in \mathbb{R}$, are solutions to the differential equation $y' = 2y + 3$.

Solution: We first compute the left-hand side of the equation.
We then compute the right-hand side of the equation.
We verify that we obtain the same expression.
The left hand side is y',

$$y'(x) = -2c\, e^{-2x}.$$

The right hand side is $2y + 3$,

$$2y + 3 = 2\Big(c\, e^{-2x} - \frac{3}{2}\Big) + 3 = (2c\, e^{-2x} - 3) + 3 = 2c\, e^{-2x}.$$

Therefore, $y' = 2y + 3$ for all $c \in \mathbb{R}$. $\lhd$

Overview of differential equations.

Remark: Differential equations have infinity many solutions.

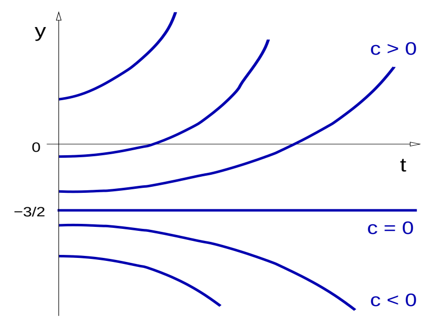
Example

For every $c \in \mathbb{R}$, the functions $y(x) = c\, e^{-2x} - \dfrac{3}{2}$ are solutions to the differential equation $y' = 2y + 3$.

The differential equation has infinitely many solutions, given by

$$y(t) = c\, e^{2t} - \frac{3}{2}, \qquad c \in \mathbb{R}.$$

- To solve a first order differential equation means to do one integration.
- So, it is reasonable that the solution contains a constant of integration, $c \in \mathbb{R}$.



Solving differential equations (Sect. 7.4)

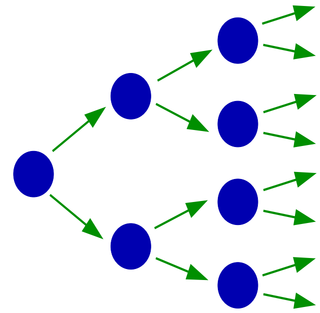
- Overview of differential equations.
- **Exponential growth.**
- Separable differential equations.

Exponential growth

Remark: The two main examples are:

(a) Population growth with unlimited food supply and no predators;

(b) Chain reactions in nuclear explosions.



- For these processes, the rate of change of a quantity y is proportional to the actual amount of that quantity.

$$\frac{dy}{dx}(x) = k\,y(x).$$

- The solution of the differential equation above is

$$y(x) = y_0\,e^{kx}, \quad y(0) = y_0.$$

Exponential growth

Example

Given a constant k, find every function y solution of the differential equation

$$\frac{dy}{dx}(x) = k\,y(x).$$

Solution: This differential equation is particularly simple to solve.

$$\frac{y'(x)}{y(x)} = k \quad \Rightarrow \quad \int \frac{y'(x)}{y(x)}\,dx = \int k\,dx.$$

Introduce the substitution $u = y(x)$, then $du = y'(x)\,dx$.

$$\int \frac{du}{u} = k \int dx \quad \Rightarrow \quad \ln(|u|) = kx + c.$$

Substitute back $y(x) = u$, and exponentiate both sides,

$$|y(x)| = e^{kx+c} = e^{kx}\,e^{c} \quad \Rightarrow \quad y(x) = \pm e^{c}\,e^{kx}.$$

Denoting $y_0 = \pm e^{c}$, we obtain $y(x) = y_0\,e^{kx}$. $\triangleleft$

Exponential growth

Example

Find every positive function y solution of the differential equation

$$\frac{dy}{dx}(x) = -3\,y(x).$$

Solution: This differential equation is particularly simple to solve.

$$\frac{y'(x)}{y(x)} = -3 \quad \Rightarrow \quad \int \frac{y'(x)}{y(x)}\,dx = \int -3\,dx.$$

Introduce the substitution $u = y(x)$, then $du = y'(x)\,dx$.

$$\int \frac{du}{u} = -3 \int dx \quad \Rightarrow \quad \ln(|u|) = -3x + c.$$

Substitute back $y(x) = u$, and exponentiate both sides,

$$|y(x)| = e^{-3x+c} = e^{-3x}\,e^{c} \quad \Rightarrow \quad y(x) = +e^{c}\,e^{-3x}.$$

Denoting $y_0 = e^c$, we obtain $y(x) = y_0\,e^{-3x}$. ◁

Solving differential equations (Sect. 7.4)

- Overview of differential equations.
- Exponential growth.
- **Separable differential equations.**

Separable differential equations

Definition

Given functions $h, g : \mathbb{R} \to \mathbb{R}$, a differential equation on the unknown function $y : \mathbb{R} \to \mathbb{R}$ is called *separable* iff the equation has the form

$$h(y)\, y'(x) = g(x).$$

Remark:

A differential equation $y'(t) = f(t, y(t))$ is separable iff

$$y' = \frac{g(x)}{h(y)} \quad \Leftrightarrow \quad f(t, y) = \frac{g(x)}{h(y)}.$$

Example

The three equations below are separable:

$$\frac{dy}{dx} = \frac{\cos(x)}{y^2}, \quad \frac{dy}{dx} = \frac{e^x}{(1+y)}, \quad 3(x+1)\, y \frac{dy}{dx} = 2(1+y^2).$$

Separable differential equations

Example

Determine whether the differential equation below is separable,

$$y'(x) = \frac{x^2}{1 - y^2(x)}.$$

Solution: The differential equation is separable, since it is equivalent to

$$(1 - y^2)\, y'(x) = x^2 \quad \Rightarrow \quad \begin{cases} g(x) = x^2, \\ h(y) = 1 - y^2. \end{cases}$$

$\lhd$

Remark: The functions g and h are not uniquely defined. Another choice here is:

$$g(x) = c\, x^2, \quad h(y) = c\,(1 - y^2), \quad c \in \mathbb{R}.$$

Separable differential equations

Example

Find every solution of the separable equation $h(y)\,y'(x) = g(x)$.

Solution: We integrate on both sides of the equation,

$$\int h(y)\,y'(x)\,dx = \int g(x)\,dx + c.$$

Introduce the substitution $u = y(x)$, then $du = y'(x)\,dx$,

$$\int h(u)\,du = \int g(x)\,dx + c.$$

Denote by H a primitive of h, that is, $H' = h$.
Denote by G a primitive of g, that is, $G' = g$.

$$H(u) = G(x) + c.$$

Substitute back $y(x) = u$,

$$H(y(x)) = G(x) + c.$$

$\triangleleft$

Separable differential equations

Example

Find every solutions of the equation $y'(x) + y^2(x)\cos(2x) = 0$.

Solution: The equation is $\dfrac{y'}{y^2} = -\cos(2x)$, separable,

$$g(x) = -\cos(2x), \qquad h(y) = \frac{1}{y^2}.$$

Integrate on both sides of the equation,

$$\frac{y'(x)}{y^2(x)} = -\cos(2x) \quad \Leftrightarrow \quad \int \frac{y'(x)}{y^2(x)}\,dx = -\int \cos(2x)\,dx + c.$$

The substitution $u = y(x)$, $du = y'(x)\,dx$, implies that

$$\int \frac{du}{u^2} = -\int \cos(2x)\,dx + c \quad \Leftrightarrow \quad -\frac{1}{u} = -\frac{1}{2}\sin(2x) + c.$$

Separable differential equations

Example

Find every solutions of the equation $y'(x) + y^2(x)\cos(2x) = 0$.

Solution: Recall: $-\frac{1}{u} = -\frac{1}{2}\sin(2x) + c$.

Substitute the function $y(x) = u$ back in the equation above,

$$-\frac{1}{y(x)} = -\frac{1}{2}\sin(2x) + c. \qquad \text{(Implicit form.)}$$

Or multiply by (-1),

$$\frac{1}{y(x)} = \frac{1}{2}\sin(2x) - c = \frac{\sin(2x) - 2c}{2}.$$

$$y(x) = \frac{2}{\sin(2x) - 2c}. \qquad \text{(Explicit form.)} \qquad \triangleleft$$

Separable differential equations

Example

From all solutions to $y'(x) + y^2(x)\cos(2x) = 0$. find the one satisfying $y(0) = 1$.

Solution: Recall: $y(x) = \frac{2}{\sin(2x) - 2c}$.

The extra condition is called the *initial condition*.

The initial condition fixes the value of the constant c.

Indeed, $1 = y(0) = \frac{2}{0 - 2c}$, so $1 = -\frac{1}{c}$, hence $c = -1$.

We conclude that $y(t) = \frac{2}{\sin(2t) + 2}$. $\triangleleft$