The exponential function (Sect. 7.3)

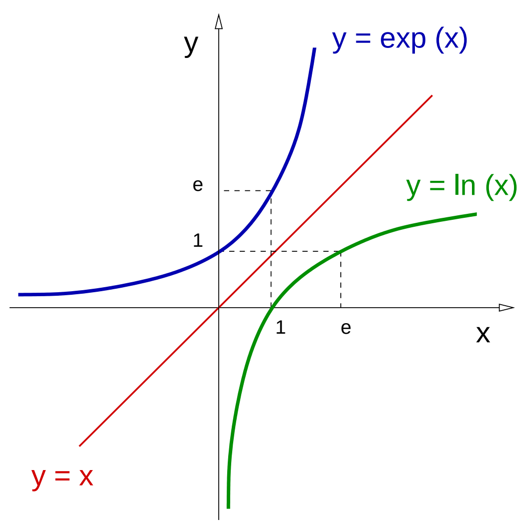
- Review: The exponential function e^x.
- Computing the number e.
- The exponential function a^x.
- Derivatives and integrals.
- Logarithms with base $a \in \mathbb{R}$.

Review: The exponential function e^x

Definition
The *exponential function*, $\exp : \mathbb{R} \to (0, \infty)$, is the inverse of the natural logarithm, that is,

$$\exp(x) = y \Leftrightarrow x = \ln(y).$$

Notation: $\exp(x) = e^x$.



Remark: Since $\ln(1) = 0$, then $e^0 = 1$.
Since $\ln(e) = 1$, then $e^1 = e$.

$$\left(e^{ax}\right)' = a\, e^{ax}, \qquad \int e^{ax}\, dx = \frac{e^{ax}}{a} + c.$$

Algebraic properties

Remark: The algebraic properties on natural logarithms translate into algebraic properties of the exponential function.

Theorem

For every a, $b \in \mathbb{R}$, and every rational number, q, hold

(a) $e^{a+b} = e^{a}, e^{b}$;

(b) $e^{-a} = \dfrac{1}{e^{a}}$;

(c) $e^{a-b} = \dfrac{e^{a}}{e^{b}}$;

(d) $(e^{a})^{q} = e^{qa}$.

Proof: Only of (a):

$$\ln\left(e^{a+b}\right) = a + b = \ln\left(e^{a}\right) + \ln\left(e^{b}\right) = \ln\left(e^{a}\,e^{b}\right).$$

We conclude that $e^{a+b} = e^{a}\,e^{b}$. □

Algebraic properties

Example

Simplify the expression $\left(\dfrac{e^{x-\ln(2)}}{e}\right)^{3}$.

Solution:

$$\left(\frac{e^{x-\ln(2)}}{e}\right)^{3} = \frac{\left(e^{x-\ln(2)}\right)^{3}}{e^{3}} = \frac{1}{e^{3}}\,e^{3x-3\ln(2)} = e^{-3}\,\frac{e^{3x}}{e^{3\ln(2)}}$$

$$\left(\frac{e^{x-\ln(2)}}{e}\right)^{3} = \frac{e^{-3}\,e^{3x}}{e^{\ln(2^{3})}} = \frac{e^{3x-3}}{e^{\ln(8)}} = \frac{e^{3(x-1)}}{8}.$$

We conclude that $\left(\dfrac{e^{x-\ln(2)}}{e}\right)^{3} = \dfrac{1}{8}\,e^{3(x-1)}$. ◁

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Computing the number e.

Theorem

The number e defined as $\ln(e) = 1$ *can be obtained as*

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

Proof: On the one hand, $\ln'(x) = \dfrac{1}{x}$, that implies $\ln'(1) = 1$.

On the other hand, $\ln'(1) = \lim_{h \to 0} \frac{1}{h}\big[\ln(1 + h) - \ln(1)\big]$, that is,

$$\ln'(1) = \lim_{h \to 0} \frac{1}{h} \ln(1 + h) = \lim_{h \to 0} \ln\big[(1 + h)^{1/h}\big].$$

The ln is continuous, $\lim_{h \to 0} \ln\big[(1 + h)^{1/h}\big] = \ln\big[\lim_{h \to 0} (1 + h)^{1/h}\big]$.

Therefore, $\ln\big[\lim_{h \to 0} (1 + h)^{1/h}\big] = 1$. But ln is a one-to-one function,

and $\ln(e) = 1$, hence $e = \lim_{h \to 0} (1 + h)^{1/h}$. $\square$

Computing the number e.

Remark: The convergence in $e = \lim_{h \to 0} (1 + h)^{1/h}$ is slow.

- For $h = 1$, $e_h = 2$.
- For $h = \frac{1}{2}$, $e_h = \big(1.5\big)^2 = 2.25$.
- For $h = \frac{1}{10}$, $e_h = \big(1.1\big)^{10} = 2.5937\dots$.
- For $h = \frac{1}{100}$, $e_h = \big(1.01\big)^{100} = 2.7048\dots$.
- For $h = \frac{1}{1000}$, $e_h = \big(1.001\big)^{1000} = 2.7169\dots$.

Remark: $e = 2.71828182\dots$.

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The exponential function a^x

Remarks:

- The exponentiation function can be generalized from base e to base $a \in (0, \infty)$.
- Recall that $a = e^{\ln(a)}$, for every $a \in (0, \infty)$.

Definition

The exponentiation function on base $a \in (0, \infty)$ is the function $\exp[a] : \mathbb{R} \to (0, \infty)$ given by

$$\exp[a](x) = e^{x \ln(a)}.$$

Remarks:

- For $a = e$ we reobtain $\exp[e](x) = e^x$.
- The exponentiation satisfies $\exp[a](0) = 1$ and $\exp[a](1) = a$.
- Also $\exp[a](m/n) = e^{(m/n)\ln(a)} = e^{\ln(a^{m/n})} = a^{m/n}$.
- Notation: $\exp[a](x) = a^x$, for $x \in \mathbb{R}$.

The exponential function a^x

Remark: The algebraic properties of e^x also hold for a^x.

Theorem

For every $a \in (0, \infty)$, b, $c \in \mathbb{R}$, and every rational number, q, hold

(a) $a^{b+c} = a^b, a^c$;

(b) $a^{-b} = \dfrac{1}{a^b}$;

(c) $a^{b-c} = \dfrac{a^b}{a^c}$;

(d) $(a^a)^q = a^{qa}$.

Proof: Only of (a):

$$a^{(b+c)} = e^{(b+c)\ln(a)} = e^{b\ln(a)+c\ln(a)} = e^{b\ln(a)}\, e^{c\ln(a)}.$$

We conclude that $a^{(b+c)} = a^b\, a^c$. □

The exponential function a^x

Example
Compute $3^{\pi+\sqrt{2}}$.

Solution:

$$3^{\pi+\sqrt{2}} = e^{(\pi+\sqrt{2})\ln(3)} = e^{(3.14...+1.41...)(1.099...)} = 149.167...$$

$\lhd$

Example
Compute $2^{-\pi}$.

Solution:

$$2^{-\pi} = \frac{1}{2^{\pi}} = \frac{1}{e^{\pi \ln(2)}} = \frac{1}{8.825...}$$

$\lhd$

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Derivatives and integrals

Theorem

For every $a \in (0, \infty)$, $c \in \mathbb{R}$, and differentiable function u holds,

$$(a^x)' = \ln(a)\, a^x, \qquad (a^u)' = \ln(a)\, a^u\, u'.$$

In addition, if $a \neq 1$, then

$$\int a^x\, dx = \frac{a^x}{\ln(a)} + c.$$

Proof of the first equation:

$$(a^x)' = \big(e^{x \ln(a)}\big)' = \ln(a)\, \big(e^{x \ln(a)}\big),$$

that is, $(a^x)' = \ln(a)\, a^x$. $\square$

Derivatives and integrals

Example

Compute both the derivative and a primitive of $f(x) = 5^x$.

Solution: The derivative is $\big(5^x\big)' = \ln(5)\, 5^x$.

The antiderivatives are $\displaystyle\int 5^x\, dx = \frac{1}{\ln(5)}\, 5^x + c$, for $c \in \mathbb{R}$. $\lhd$

Example

Compute both the derivative and a primitive of $f(x) = 5^{3x}$.

Solution: $\big(5^{3x}\big)' = \ln(5)\, 5^{3x} (3x)'$, hence $\big(5^{3x}\big)' = 3 \ln(5)\, 5^{3x}$,

For the antiderivatives use $u = 3x$, $du = 3\, dx$,

$$I = \int 5^{3x}\, dx = \int 5^u\, \frac{du}{3} = \frac{1}{3}\, \frac{5^u}{\ln(5)} \quad \Rightarrow \quad I = \frac{5^{3x}}{3 \ln(5)} + c.$$

$\lhd$

Derivatives and integrals

Example

Compute $I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\,dx$.

Solution: Use the substitution $u = \sin(x)$, then $du = \cos(x)\,dx$.

$$I = \int \left(\frac{1}{7}\right)^{\sin(x)} \cos(x)\,dx = \int \left(\frac{1}{7}\right)^{u} du$$

$$I = \frac{1}{\ln(1/7)} \left(\frac{1}{7}\right)^{u} + c.$$

Now substitute back,

$$I = -\frac{1}{\ln(7)} \left(\frac{1}{7}\right)^{\sin(x)} + c. \qquad \triangleleft$$

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Logarithms with base $a \in \mathbb{R}$.

Remarks:

- The function $a^x = e^{x\ln(a)}$ is one-to-one, so invertible.
- $\log_a(x)$, a logarithm with base a, is the inverse of a^x.
- The function $\log_a$ is proportional to ln.

Definition

For every positive a with $a \neq 1$ the function $\log_a : (0, \infty) \to \mathbb{R}$ is given by

$$\log_a(x) = y \quad \Leftrightarrow \quad x = a^y.$$

Theorem

For positive a with $a \neq 1$ holds $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$.

Proof: $\log_a(x) = y \Leftrightarrow x = a^y = e^{y\ln(a)} \Leftrightarrow \ln(x) = y\ln(a)$.

Therefore, $\ln(x) = \log_a(x)\ln(a) \quad \Rightarrow \quad \log_a(x) = \dfrac{\ln(x)}{\ln(a)}$. □

Logarithms with base $a \in \mathbb{R}$.

Theorem

For every positive a, $a \neq 1$, and differentiable function u holds,

$$\log_a'(x) = \frac{1}{\ln(a)\,x}, \qquad \big[\log_a(u)\big]' = \frac{u'}{\ln(a)\,u}.$$

Proof of the first equation: Since $\log_a(x) = \dfrac{\ln(x)}{\ln(a)}$, then

$$\log_a'(x) = \frac{1}{\ln(a)}\,\frac{1}{x}$$

□

Example

Compute the derivative of $f(x) = \log_2(3x^3 + 2)$.

Solution: $f'(x) = \dfrac{1}{\ln(2)}\,\ln'(3x^2 + 2) = \dfrac{1}{\ln(2)}\,\dfrac{1}{(3x^2 + 2)}\,6x$.

We conclude: $f'(x) = \dfrac{6x}{\ln(2)\,(3x^2 + 2)}$. ◁