The exponential function (Sect. 7.3)

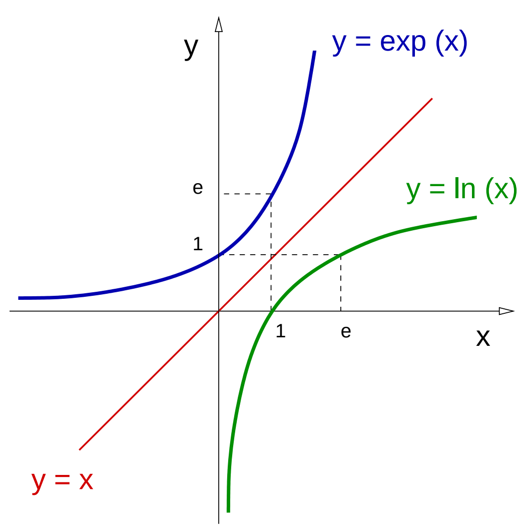
- The inverse of the logarithm.
- Derivatives and integrals.
- Algebraic properties.

The inverse of the logarithm

Remark: The natural logarithm $\ln : (0, \infty) \to \mathbb{R}$ is a one-to-one function, hence invertible.

Definition
The *exponential function*, $\exp : \mathbb{R} \to (0, \infty)$, is the inverse of the natural logarithm, that is,

$$\exp(x) = y \Leftrightarrow x = \ln(y).$$



Remark: Since $\ln(1) = 0$, then $\exp(0) = 1$.
Since $\ln(e) = 1$, then $\exp(1) = e$.

The inverse of the logarithm

Remark: Since $\ln\big(e^{m/n}\big) = \frac{m}{n}\ln(e) = \frac{m}{n}$, then holds

$$\exp\Big(\frac{m}{n}\Big) = e^{m/n}$$

The exponentiation of a rational number is the power function.

The exponentiation is a way to extend the power function from rational numbers to irrational numbers.

Definition

For every $x \in \mathbb{R}$ we denote $e^x = \ln^{-1}(x) = \exp(x)$.

Example

Find x solution of $e^{3x+1} = 2$.

Solution: Compute ln on both sides,

$$\ln\big(e^{3x+1}\big) = \ln(2) \Rightarrow 3x + 1 = \ln(2) \Rightarrow x = \frac{1}{3}\big[\ln(2) - 1\big]. \quad \triangleleft$$

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Derivatives and integrals

Remark: The derivative of the exponential is the same exponential.

Theorem (Derivative of the exponential)

(a) *For every* $x \in \mathbb{R}$ *holds* $\big(e^x\big)' = e^x$.

(b) *For every differentiable function* u *holds* $\big(e^u\big)' = e^u\, u'$.

Proof:

(a) $$\ln\big(e^x\big) = x \quad \Rightarrow \quad \frac{d}{dx}\ln\big(e^x\big) = 1 \quad \Rightarrow \quad \frac{1}{e^x}\,\big(e^x\big)' = 1$$

We conclude that $\big(e^x\big)' = e^x$.

(b) Chain rule implies

$$\big(e^u\big)' = \frac{de^u}{du}\,u' \quad \Rightarrow \quad \big(e^u\big)' = e^u\,u'.$$

$\square$

Remark: In particular: $\big(e^{ax}\big)' = a\,e^{ax}$, for $a \in \mathbb{R}$.

Derivatives and integrals

Remark:
Part (a) of the Theorem can be proven with the formula
$\big(f^{-1}\big)'(x) = \dfrac{1}{f'\big(f^{-1}(x)\big)}$, for $f^{-1}(x) = e^x$. Indeed,

$$\big(e^x\big)' = \frac{1}{\ln'\big(e^x\big)}, \quad \frac{d\ln}{dy}(y) = \frac{1}{y} \quad \Rightarrow \quad \big(e^x\big)' = e^x.$$

Example
Find y' for $y(x) = e^{(3x^2+5)}$.

Solution: We use all the well-known derivation rules,

$$y' = \big(e^{(3x^2+5)}\big)' = e^{(3x^2+5)}\,(3x^2+5)' \quad \Rightarrow \quad y' = 6x\,e^{(3x^2+5)}.$$

$\triangleleft$

Derivatives and integrals

Example

Find y' for $y(x) = e^{\sin(3x^2)} \ln(x^2+1)$.

Solution: We start with the product rule,

$$y' = \left(e^{\sin(3x^2)}\right)' \ln(x^2+1) + e^{\sin(3x^2)} \left(\ln(x^2+1)\right)'$$

$$y' = e^{\sin(3x^2)} \left(\sin(3x^2)\right)' \ln(x^2+1) + e^{\sin(3x^2)} \frac{1}{(x^2+1)} (x^2+1)'$$

$$y' = e^{\sin(3x^2)} \cos(3x^2)\,(6x) \ln(x^2+1) + e^{\sin(3x^2)} \frac{(2x)}{(x^2+1)}.$$

$$y' = 2x\, e^{\sin(3x^2)} \left[3\cos(3x^2)\, \ln(x^2+1) + \frac{1}{(x^2+1)}\right]. \qquad \triangleleft$$

Derivatives and integrals

Remark: The derivation rule for the exponential implies that its antiderivative is

$$\int e^{ax}\, dx = \frac{e^{ax}}{a} + c.$$

Example

Find $I = \displaystyle\int_0^{\pi/4} e^{3\sin(2x)} \cos(2x)\, dx$.

Solution: Use the substitution $u = 3\sin(2x)$, $du = 6\cos(2x)\, dx$.

$$I = \int_0^{\pi/4} e^{3\sin(2x)} \cos(2x)\, dx = \int_0^1 e^u \frac{du}{6} = \frac{1}{6}\left(e^u\right)\Big|_0^1.$$

Since, $I = \dfrac{1}{6}(e^1 - e^0)$, we obtain $I = \dfrac{1}{6}(e-1)$. $\triangleleft$

Derivatives and integrals

Example

Find $I = \int 3x\, e^{2x^2} \sin\big(e^{2x^2}\big)\, dx$.

Solution: Recall that $\big(e^{2x^2}\big)' = (2x^2)'\, e^{2x^2} = 4x\, e^{2x^2}$.

Therefore, use the substitution $u = e^{2x^2}$, since $\dfrac{du}{4} = x\, e^{2x^2}\, dx$,

$$I = \int 3\frac{du}{4} \sin(u) = \frac{3}{4} \int \sin(u)\, du = \frac{3}{4} \big(-\cos(u)\big)$$

Substitute back the original unknown,

$$I = -\frac{3}{4} \cos\big(e^{2x^2}\big). \qquad \triangleleft$$

Derivatives and integrals

Example

Find the solution to the initial value problem

$$y''(x) = 18\, e^{3x}, \quad y(0) = 1, \quad y'(0) = 2$$

Solution: We first find y', integrating the equation above,

$$\int y''(x)\, dx = \int 18\, e^{3x}\, dx + c \quad \Rightarrow \quad y' = \frac{18}{3}\, e^{3x} + c = 6\, e^{3x} + c.$$

The initial condition fixes c,

$$2 = y'(0) = 6\, e^{0} + c \quad \Rightarrow \quad c = -4 \quad \Rightarrow \quad y'(x) = 6\, e^{3x} - 4.$$

We now need to integrate one more time.

Derivatives and integrals

Example

Find the solution to the initial value problem

$$y''(x) = 18\, e^{3x}, \quad y(0) = 1, \quad y'(0) = 2$$

Solution: Recall: $y'(x) = 6\, e^{3x} - 4$.

$$\int y'(x)\, dx = \int \big(6\, e^{3x} - 4\big)\, dx + c = 6 \int e^{3x}\, dx - 4 \int dx + c$$

$$y(x) = \frac{6}{3}\, e^{3x} - 4x + c = 2\, e^{3x} - 4x + c.$$

The initial condition implies

$$1 = y(0) = 2\, e^{0} + c \quad \Rightarrow \quad c = -1.$$

We conclude that $y(x) = 2\, e^{3x} - 4x - 1$. $\triangleleft$

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Algebraic properties

Remark: The algebraic properties on natural logarithms translate into algebraic properties of the exponential function.

Theorem

For every a, b, c $\in \mathbb{R}$, and every rational number, q, hold

(a) $e^{a+b} = e^a, e^b$;

(b) $e^{-a} = \dfrac{1}{e^a}$;

(c) $e^{a-b} = \dfrac{e^a}{e^b}$;

(d) $\left(e^a\right)^q = e^{qa}$.

Proof: Only of (a):

$$\ln\left(e^{a+b}\right) = a + b = \ln\left(e^a\right) + \ln\left(e^b\right) = \ln\left(e^a\, e^b\right).$$

We conclude that $e^{a+b} = e^a\, e^b$. □

Algebraic properties

Example

Simplify the expression $\left(\dfrac{e^{x-\ln(2)}}{e}\right)^3$.

Solution:

$$\left(\frac{e^{x-\ln(2)}}{e}\right)^3 = \frac{\left(e^{x-\ln(2)}\right)^3}{e^3} = \frac{1}{e^3}\, e^{3x-3\ln(2)} = e^{-3}\, \frac{e^{3x}}{e^{3\ln(2)}}$$

$$\left(\frac{e^{x-\ln(2)}}{e}\right)^3 = \frac{e^{-3}\, e^{3x}}{e^{\ln(2^3)}} = \frac{e^{3x-3}}{e^{\ln(8)}} = \frac{e^{3(x-1)}}{8}.$$

We conclude that $\left(\dfrac{e^{x-\ln(2)}}{e}\right)^3 = \dfrac{1}{8}\, e^{3(x-1)}$. ◁