The inverse function (Sect. 7.1)

- One-to-one functions.
- The inverse function
- The graph of the inverse function.
- Derivatives of the inverse function.

One-to-one functions

Remark:

- Not every function is invertible.
- Only one-to-one functions are invertible.

Definition

A function $f : D \to \mathbb{R}$ is called *one-to-one* (injective) iff for every $x_1, x_2 \in D$ holds

$$x_1 \neq x_2 \quad \Rightarrow \quad f(x_1) \neq f(x_2).$$

Example

Invertible:

1. $y = x^3$, for $x \in \mathbb{R}$.
2. $y = x^2$, for $x \in [0, b]$.
3. $y = \sqrt{x}$, for $x \in [0, \infty)$.
4. $y = \sin(x)$, $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

Not Invertible:

1. $y = x^2$, for $x \in [-a, a]$.
2. $y = |x|$, for $x \in [-a, a]$.
3. $y = \cos(x)$, $x \in [-a, a]$.
4. $y = \sin(x)$, for $x \in [0, \pi]$.

One-to-one functions

Example

Verify that the functions below are not one-to-one:

(a) $y = x^2$, for $x \in [-a, a]$.

(b) $y = \cos(x)$, $x \in [-a, a]$.

(c) $y = \sin(x)$, for $x \in [0, \pi]$.

Solution:

(a) For $x_1 = -1$, $x_2 = 1$ we have that $x_1 \neq x_2$ and

$$f(x_1) = (-1)^2 = 1 = 1^2 = f(x_2).$$

(b) Recalling that $\cos(\theta) = \cos(-\theta)$, and taking $x_1 = -\theta$, $x_2 = \theta$,

$$f(-\theta) = \cos(-\theta) = \cos(\theta) = f(\theta).$$

(c) Since $\sin(\theta) = \sin(\pi - \theta)$, and taking $x_1 = \pi/4$, $x_2 = 3\pi/4$,

$$f(\pi/4) = \sin(\pi/4) = \sin(\pi - \pi/4) = \sin(3\pi/4) = f(3\pi/4).$$ ◁

One-to-one functions

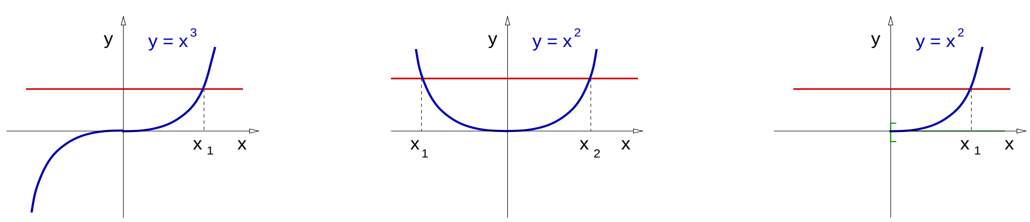
Remark: By looking at the graph of the function one can determine whether the function is one-to-one or not.

Theorem (Horizontal line test)

A function $f : D \to R$ is one-to-one iff the function graph intersects every horizontal line at most once.

Proof: If a function f intersects the horizontal line $y = y_0$ at $(x_1, f(x_1))$ and $(x_2, f(x_2))$, with $x_1 \neq x_2$, that means $y_0 = f(x_1) = f(x_2)$. Hence f is not one-to-one. □

Example



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The inverse function

Remark: Only one-to-one functions are invertible.

Definition

The *inverse* of a one-to-one function $f : D \to R$ is the function $f^{-1} : R \to D$ defined for all $x \in D$ and all $y \in R$ as follows

$$f^{-1}(y) = x \quad \Leftrightarrow \quad y = f(x).$$

Example

Find the inverse of $f(x) = 2x - 3$.

Solution: Denote $y = f(x)$, that is, $y = 2x - 3$. Find x in the expression above,

$$2x = y + 3 \quad \Rightarrow \quad x = \frac{1}{2}y + \frac{3}{2}.$$

Then, the inverse function is $f^{-1}(y) = \frac{1}{2}y + \frac{3}{2}$. $\triangleleft$

The inverse function

Remark:

- If f^{-1} is the inverse of f, then holds

$$(f^{-1} \circ f)(x) = x, \quad (f \circ f^{-1})(y) = y.$$

- Equivalently,

$$f^{-1}\big(f(x)\big) = x, \quad f\big(f^{-1}(y)\big) = y.$$

Example

Verify the relations above for $f(x) = 2x - 3$.

Solution: Recall: $f^{-1}(y) = (y+3)/2$. Hence

$$f^{-1}(f(x)) = f^{-1}(2x-3) = \frac{1}{2}\big[(2x-3)+3\big] = x.$$

$$f(f^{-1}(y)) = f\Big(\frac{1}{2}(y+3)\Big) = 2\left[\frac{1}{2}(y+3)\right] - 3 = y. \qquad \triangleleft$$

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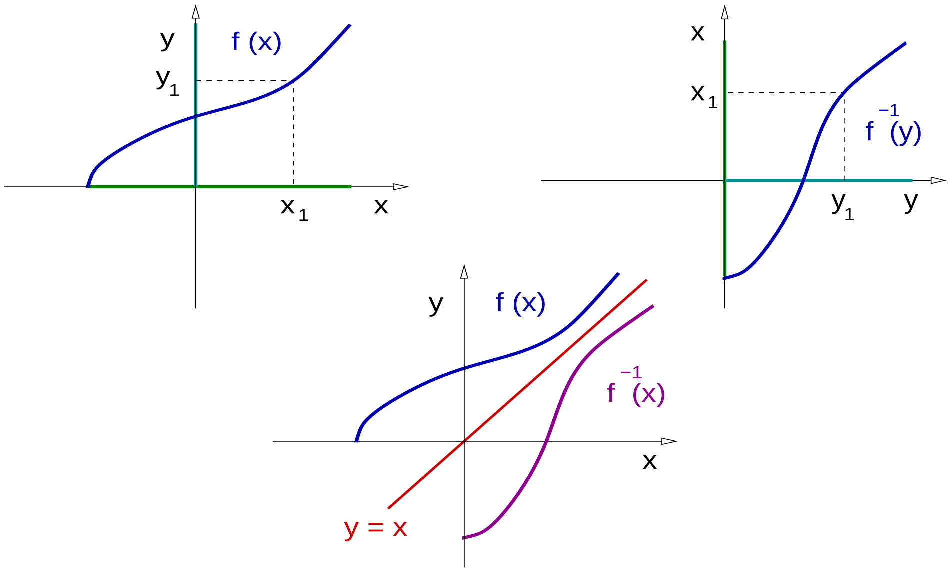
The graph of the inverse function

Remark: The graph of the function f^{-1} is obtained reflecting the graph of f along the line $y = x$.

Example



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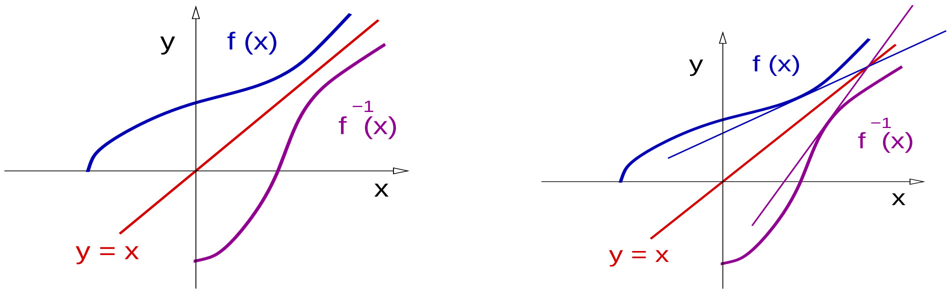
Derivatives of the inverse function.

Remark: The derivative values of a function and its inverse are deeply related.



Theorem (Derivative for inverse functions)

If the invertible function $f : D \to R$ is differentiable and $f'(x) \neq 0$ for every $x \in D$, then the function $f^{-1} : R \to D$ is differentiable. Furthermore, for every $y \in R$ holds

$$\big(f^{-1}\big)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

Derivatives of the inverse function.

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$$\big(f^{-1}\big)'(y) = \frac{1}{f'\big(f^{-1}(y)\big)}.$$

Example

Verify the Theorem above for $f(x) = 2x - 3$.

Solution: This case is simple because $f'(x) = 2$, constant.

Since $f^{-1}(y) = \frac{1}{2}(y + 3)$, then $\big(f^{-1}\big)'(y) = \frac{1}{2}$, constant.

Therefore, $\big(f^{-1}\big)' = \frac{1}{f'}$. $\lhd$

Derivatives of the inverse function.

Example

Verify the Theorem above for $f(x) = x^3$.

Solution:

We first compute the inverse function. Denote $y = f(x)$, then

$$y = x^3 \quad \Rightarrow \quad x = y^{1/3} \quad \Rightarrow \quad f^{-1}(y) = y^{1/3}.$$

Compute now the derivative of the inverse function,

$$\big(f^{-1}\big)'(y) = \frac{1}{3}\, y^{-2/3} = \frac{1}{3}\,\frac{1}{y^{2/3}}.$$

Compute now the derivative of the original function,

$$f'(x) = 3\,x^2 \quad \Rightarrow \quad f'\big(f^{-1}(y)\big) = 3\big(f^{-1}(y)\big)^2 = 3\big(y^{1/3}\big)^2.$$

We conclude that $\dfrac{1}{3\,y^{2/3}} = \big(f^{-1}\big)'(y) = \dfrac{1}{f'\big(f^{-1}(y)\big)}$. ◁

Derivatives of the inverse function.

Example

Verify the Theorem above for $f(x) = x^3 + x - 1$ at $x = 2$.

Solution: In this case is difficult to find $f^{-1}(y)$ for $y \in \mathbb{R}$.

But at $x = 2$ we have $f(2) = 2^3 + 2 - 1$, that is, $f(2) = 9$.

Therefore $f^{-1}(9) = 2$.

Compute now the derivative of the original function,

$$f'(x) = 3\,x^2 + 1.$$

Then the formula in the Theorem, $\big(f^{-1}\big)'(y) = \dfrac{1}{f'\big(f^{-1}(y)\big)}$,

for $y = 9$ implies

$$\big(f^{-1}\big)'(9) = \frac{1}{f'\big(f^{-1}(9)\big)} = \frac{1}{f'(2)} = \frac{1}{3\,(4) + 1}.$$

We conclude that $\big(f^{-1}\big)'(y) = \dfrac{1}{13}$. ◁