Work on solids and fluids (Sect. 6.5)

- Moving things around.
- Forces made by springs.
- Pumping liquids.

Moving Things around: Constant forces

Remarks:

- Moving things around requires some work.
- Work is an amount of energy needed to move an object.

Remark: If an object is moved a distance d along a straight line by a constant force F in the direction of motion, then the work done on the particle is

$$W = Fd.$$

Example

Find the work done to lift an object with mass of $m = 20\ Kgr$ from the ground to a height of $d = 1\ ft$.

Solution: The force to lift the object is $F = mg$, with $g \simeq 10\ \dfrac{m}{s^2}$.

Then, $F = (20)\ Kgr\ (10)\ \dfrac{m}{s^2} = 200\ N$, so

$$W = Fd = 200\ N\ \frac{3}{10}\ m \quad \Rightarrow \quad W = 60\ J.$$

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Moving Things around: Variable forces

Definition

The *work* done on a particle moving on the x-axis by a non-constant force F along the x-axis for $x \in [a, b]$ is

$$W = \int_a^b F(x)\,dx.$$

Remarks:

- The formula above is obtained in the standard way: Introduce a partition in $[a, b]$ and compute the limit of partial sums
$$W_N = \sum_{k=0}^{N-1} F(x_k)\,\Delta x_k.$$
- The simplest variable force is the one produced by a spring. In 1660 Robert Hooke discovered that $F(x) = kx$, where k is called the spring constant, and x is the displacement from the spring rest position.

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Forces made by springs

Remark: The force of a spring, $F(x) = k\,x$ is called *Hooke's Law*.

Example

Find the minimum work needed to compress a spring with constant $k = 3\ N/m$ a distance of $d\ m$ from the spring rest position.

Solution: The spring force is $F(x) = kx$, then

$$W = \int_0^d kx\,dx = k\frac{x^2}{2}\Big|_0^d \quad \Rightarrow \quad W = \frac{kd^2}{2}. \qquad \triangleleft\,.$$

Example

If a force of 60 N stretches a spring 3 m from its rest position, how much work does it take to stretch it 4 m from its rest position?

Solution: From Hooke's Law we know that $60\ N = k\ (3)\ m$, that is, $k = 20\ N/m$. The previous problem implies $W = kd^2/2$, that is,

$$W = 20\ \frac{N}{m}\ \frac{4^2}{2}\ m^2 \quad \Rightarrow \quad W = 160\ J. \qquad \triangleleft$$

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Pumping liquids

Remark: Pumping liquids in or out an arbitrary shaped container is a typical problem with variable forces.

Theorem

Consider an arbitrary shaped container with horizontal cross section area $A(z)$, for $z \in [0, h]$, and let $g = 9.81\ m/s^2$.

(a) *If a liquid of density δ Kgr/m^3 is resting at the bottom of the container, then the work done to pump the liquid in the container, initially empty, up to a height $h_1 \leqslant h$ is*

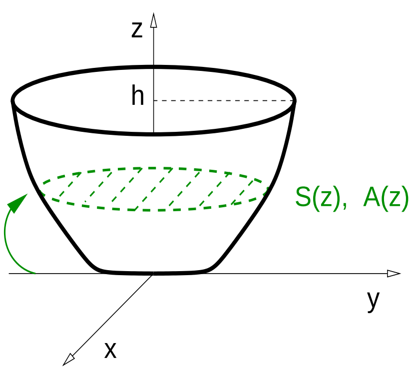
$$W = \int_0^{h_1} \delta\, g\, A(z)\, z\, dz.$$

(b) *The work done to pump the liquid out from the top of a container, initially filled with liquid up to a height $h_1 \leqslant h$ is*

$$W = \int_0^{h_1} \delta\, g\, A(z)\, (h - z)\, dz.$$

Pumping liquids

Proof: (a) Show: $W = \displaystyle\int_0^{h_1} \delta\, g\, A(z)\, z\, dz.$



The amount of liquid that can be placed at cross-section $S(z)$ is

$$L = \delta\, A(z)\, dz.$$

The force that must be done to lift that amount of liquid is

$$F = \delta\, g\, A(z)\, dz.$$

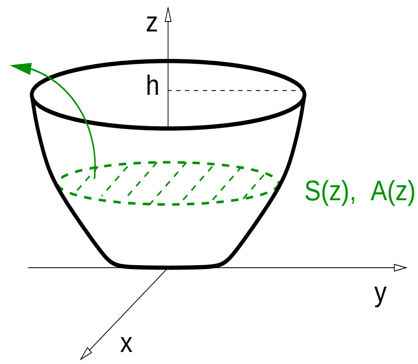
The work done to lift that liquid to height z from $z = 0$ is

$$W(z) = \delta\, g\, A(z)\, z\, dz.$$

The work to fill in the container up to h_1 is $W = \displaystyle\int_0^{h_1} \delta\, g\, A(z)\, z\, dz.$

Pumping liquids

Proof: (b) Show: $W = \int_0^{h_1} \delta\, g\, A(z)\, (h - z)\, dz$.



The force that must be done to lift the liquid in $S(z)$ is

$$F = \delta\, g\, A(z)\, dz.$$

The work done to lift that liquid from a height z to h is

$$W(z) = \delta\, g\, A(z)\, (h - z)\, dz.$$

The work to empty the container initially filled up to h_1 is

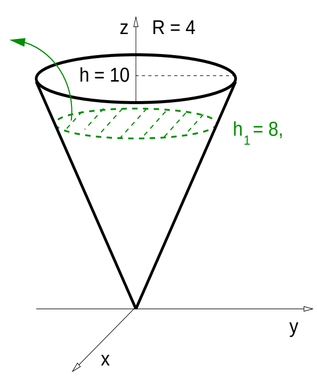
$$W = \int_0^{h_1} \delta\, g\, A(z)\, (h - z)\, dz.$$

Pumping liquids

Example

A tank has the shape of an inverted circular cone with height $h = 10\ m$ and base radius $R = 4\ m$. It is filled with water to a height $h_1 = 8\ m$. Recalling that the water density is $1\ gr/cm^3 = 1000\ Kgr/m^3$, find the work required to empty the tank pumping the water from the top.

Solution:



Recall: $W = \int_0^{h_1} \delta\, g\, A(z)\, (h - z)\, dz$.

Here $A(z) = \pi \big[R(z)\big]^2 = \pi \big[y(z)\big]^2$.

$z(y) = \dfrac{10}{4}\, y$, so $y = \dfrac{2}{5}\, z$. Hence

$$W = \delta g \pi \frac{4}{25} \int_0^8 z^2 (10 - z)\, dz.$$

Pumping liquids

Example

A tank has the shape of an inverted circular cone with height $h = 10\ m$ and base radius $R = 4\ m$. It is filled with water to a height $h_1 = 8\ m$. Recalling that the water density is $1\ gr/cm^3 = 1000\ Kgr/m^3$, find the work required to empty the tank pumping the water from the top.

Solution: Recall: $W = \delta g \pi \frac{4}{25} \int_0^8 z^2 (10 - z)\, dz.$

$$W = \delta g \pi \frac{4}{25} \Big[10\, \frac{z^3}{3}\Big|_0^8 - \frac{z^4}{4}\Big|_0^8 \Big] = \delta g \pi \frac{4}{25}\, 8^3 \Big[\frac{10}{3} - \frac{8}{4} \Big]$$

$$W = \delta g \pi \frac{4}{25}\, 8^3\, \frac{4}{3} \quad \Rightarrow \quad W = \delta g \pi \frac{16}{25}\, 8^3.$$

That is, $W = 3.4\, x\, 10^6\ J$. ◁